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On Market Games[1]

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The "market games"—games that derive from an exchange economy in which the traders have continuous concave monetary utility functions, are shown to be the same as the "totally balanced games"—games which with all their subgames possess cores. (The core of a game is the set of outcomes that no coalition can profitably block.) The coincidence of these two classes of games is established with the aid of explicit transformations that generate a game from a market and vice versa. It is further shown that any game with a core has the same *solutions*, in the von Neumann-Morgenstern sense, as some totally balanced game. Thus, a market may be found that reproduces the solution behavior of any game that has a core. In particular, using a recent result of Lucas, a ten-trader ten-commodity market is described that has no solution.

1. INTRODUCTION

Recent discovery of n-person games in the classical theory which either possess no solutions [*8*, *9*], or have unusually restricted classes of solutions [*6*, *7*, *10*, *18*], has raised the question of whether these games are mere mathematical curiosities or whether they could actually arise in application. Since the most notable applications of n-person game theory to date have been to economic models of exchange, or exchange and production [*3*, *13*, *15*, *16*, *19–23*], the question may be put in a more concrete form: Are there markets, or other basic economic systems, that when interpreted as n-person games give rise to the newly-discovered counter-

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examples? If so, can they be distinguished from the ordinary run of market models, on some economic, heuristic, or even formal grounds —i.e. do they give any advance warning of their peculiar solution properties?

These questions stimulated the present investigation. The answers are "bad news": Yes, the games can arise in economics; No, there are no outwardly distinguishing features. In reaching these conclusions, however, we were led to a positive result: a surprisingly simple mathematical criterion that tells precisely which games can arise from economic models of exchange (with money). In fact, this criterion identifies a very fundamental class of games, called "totally balanced", whose further study seems merited quite apart from any consideration of solution abnormalities. Of technical interest, our derivation of the basic properties of these games and their solutions makes a substantial application of the recently-developed theory of *balanced sets* [*1*, *2*, *11*, *17*], as well as of the older work of Gillies on *domination-equivalence* [*4*, *5*].

In the present note we confine our attention to the classical theory "with side payments" [*23*]; this corresponds in the economic interpretation to the assumption that an ideal money, free from income effects or transfer costs, is available.[2] We further restrict ourselves to exchange economies, without explicit production or consumption processes, in which the commodities are finite in number and perfectly divisible and transferable, and in which the traders, also finite in number, are motivated only by their own final holdings of goods and money, their utility functions being continuous and concave and additive in the money term.

For our immediate purpose, these strictures do not matter, since the anomalous games are already attainable within the limited class of ideal markets considered. But for our larger purpose—that of initiating a systematic study of "market games" as distinct from games in general—some relaxation may be desirable, particularly with regard to money. The propects for significant generalizations in this direction appear good, and we intend to pursue them in subsequent work.

1.1 Outline of the Contents

The notions of game, core, and balanced set are reviewed. A game is called "balanced" if it has a core, and "totally balanced" if all of the subgames obtained by restricting the set of players have cores as well (Section 2).

A "market" is defined as an exchange economy with money, in which the traders have utility functions that are continuous and concave. The method of passing from a market to its "market game" is described.

[2] For a discussion of this assumption, see [*20*], pp. 807–808.

The market games with n traders form a closed convex cone in the space of all n-person games with side payments. Every market game not only has a core, but is totally balanced (Section 3).

A canonical market form—the "direct market"—is introduced, in which the commodities are in effect the traders themselves, made infinitely divisible, and the utility functions are all the same and are homogeneous of degree one. The method of passing from any game to its direct market is described, the utilities being based upon the optimal assignment of "fractional players" to the various coalitional activities. The "cover" of a game is defined as the market game of its direct market; the cover is at least as profitable to all coalitions as the original game. Every totally balanced game is its own cover, and hence is a market game. This shows that *the class of market games and the class of totally balanced games are the same.* Moreover, every market is game-theoretically equivalent to a direct market (Section 4).

The notions of imputation, domination, and solution are reviewed. Games are "*d*-equivalent" if they have identical domination relations on identical imputation spaces. They therefore have identical solutions (or lack of solutions), and their cores, if any, are the same. It is shown that *every balanced game is* d-*equivalent to a totally balanced game.* Hence, for every game with a core, there is a market that has precisely the same set of solutions (Section 5).

Using Lucas's solutionless game [*8*, *9*], a direct market is constructed that has ten traders and ten commodities (plus money), and that has no solution. Another version in the form of a production economy is also presented. Several other examples of market games with unusual solution properties are mentioned, and in one case, where the solutions contain arbitrary components, the utility function is worked out explicitly (Section 6).

2. GAMES AND CORES

For the purpose of this note, a *game* is an ordered pair $(N; v)$, where N is a finite set [the players] and v is a function from the subsets of N [coalitions] to the reals satisfying $v(O) = 0$, called the *characteristic function.* A *payoff vector* for $(N; v)$ is a point α in the $|N|$-dimensional vector space E^N whose coordinates α^i are indexed by the elements of N. If $\alpha \in E^N$ and $S \subseteq N$, we shall write $\alpha(S)$ as an abbreviation for $\sum_{i \varepsilon S} \alpha^i$.

The *core* of $(N; v)$ is the set of all payoff vectors α, if any, such that

$$\alpha(S) \geqq v(S), \quad \text{all } S \subseteq N, \tag{2-1}$$

and

$$\alpha(N) = v(N). \tag{2-2}$$

If no such α exists, we shall say that $(N; v)$ *has no core*. (Thus, in this usage, the core may be nonexistent, but is never empty.)

2.1 Balanced Sets of Coalitions

A *balanced set* $\mathscr{B}$ is defined to be a collection of subsets S of N with the property that there exist positive numbers γ_S, $S \in \mathscr{B}$, called "weights", such that for each $i \in N$ we have

$$\sum_{\substack{S \in \mathscr{B} \\ S \ni i}} \gamma_S = 1. \tag{2-3}$$

If all $\gamma_S = 1$, we have a partition of N; thus, balanced sets may be regarded as generalized partitions.

For example, if $N = \overline{1234}$, then $\{\overline{12}, \overline{13}, \overline{14}, \overline{234}\}$ is a balanced set, by virtue of the weights 1/3, 1/3, 1/3, 2/3.

A game $(N; v)$ is called *balanced* if

$$\sum_{S \in \mathscr{B}} \gamma_S v(S) \leqq v(N) \tag{2-4}$$

holds for every balanced set $\mathscr{B}$ with weights $\{\gamma_S\}$.[3]

Theorem 1. *A game has a core if and only if it is balanced.*

This is proved in [*17*]. In Scarf's generalization to games without transferable utility [*11*], all balanced games have cores, but some games with cores are not balanced. If our present results can be generalized in this direction, we conjecture that it will be the balance property, rather than the core property, that plays the central role.

2.2 Totally Balanced Games

By a *subgame* of $(N; v)$ we shall mean a game $(R; v)$ with $O \subset R \subseteq N$. Here v is the same function, but implicitly restricted to the domain consisting of the subsets of R. A game will be said to be *totally balanced* if all of its subgames are balanced. In other words, all subgames of a totally balanced game have cores.

Not all balanced games are totally balanced. For example, let $N = \overline{1234}$ and define $v(S) = 0, 0, 1, 2$ for $|S| = 0, 1, 3, 4$ respectively, and, for $|S| = 2$:

$$v(\overline{12}) = v(\overline{13}) = v(\overline{23}) = 1$$
$$v(\overline{14}) = v(\overline{24}) = v(\overline{34}) = 0.$$

[3] These conditions are heavily redundant; it suffices to assert (2-4) for the *minimal* balanced sets $\mathscr{B}$ (which moreover have unique weights). In the case of a superadditive game, only the minimal balanced sets that contain no disjoint elements are needed. (See [*17*].)

This game has a core, including the vector $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ among others. But it is not totally balanced, since the subgame $(\overline{123}; v)$ has no core.

3. MARKETS AND MARKET GAMES

For the purpose of this note, a *market* is a special mathematical model, denoted by the symbol (T, G, A, U). Here T is a finite set [the traders]; G is the nonnegative orthant of a finite-dimensional vector space [the commodity space]; $A = \{a^i: i \in T\}$ is an indexed collection of points in G [the initial endowments]; and $U = \{u^i: i \in T\}$ is an indexed collection of continuous, concave functions from G to the reals [the utility functions]. When we wish to indicate that $u^i \equiv u$, all $i \in T$ [the special case of "equal tastes"], we shall sometimes denote the market by the more specific symbol $(T, G, A, \{u\})$.

If S is any subset of T, an indexed collection $X^S = \{x^i: i \in S\} \subset G$ such that $\Sigma_S x^i = \Sigma_S a^i$ will be called a *feasible S-allocation* of the market (T, G, A, U).

A market (T, G, A, U) can be used to "generate" a game $(N; v)$ in a natural way. We set $N = T$, and define v by

$$v(S) = \max_{X^S} \sum_{i \in S} u^i(x^i), \quad \text{all } S \subseteq N, \tag{3-1}$$

where the maximum runs over all feasible S-allocations. Any game that can be generated in this way from some market is called a *market game.*[4]

In the special case of identical utility functions $u^i \equiv u$, we have

$$v(S) = |S| u(\textstyle\sum_S a^i/|S|), \quad \text{all } S \subseteq N; \tag{3-2}$$

this is a simple consequence of concavity. In the still more special case where u is homogeneous of degree 1, we have simply

$$v(S) = u(\textstyle\sum_S a^i), \quad \text{all } S \subseteq N. \tag{3-3}$$

3.1 Some Elementary Properties

The following two theorems are of a routine nature; they show that the property of being a market game is invariant under "strategic equivalence", and that the set of all market games on N forms a convex cone in the $(2^{|N|}-1)$-dimensional space of all games on N.

THEOREM 2. *If $(N; v)$ is a market game, if $\lambda \geqq 0$, and if c is an additive set function on N, then $(N; \lambda v + c)$ is a market game.*

Proof. We need merely take any market that generates $(N; v)$ and replace each utility function $u^i(x)$ by $\lambda u^i(x) + c(\{i\})$. Q.E.D.

[4] For examples, see [*13, 15, 16, 20, 21, 22*]. The abstract definition of "market game" proposed in [*13*] is not equivalent to the present one, however.

THEOREM 3. *If* $(N; v')$ *and* $(N; v'')$ *are market games, then* $(N, v'+v'')$ *is a market game.*

Proof. Let (N, G', A', U') and (N, G'', A'', U'') be markets that generate $(N; v')$ and $(N; v'')$ respectively. We shall superimpose these two markets, keeping the two sets of commodities distinct. Specifically, let G be the set of all ordered pairs (x', x'') of points from G' and G'' respectively; let A be the set of pairs (a'^i, a''^i) of correspondingly-indexed elements of A' and A''; and let U be the set of sums:

$$u^i((x', x'')) = u'^i(x') + u''^i(x'')$$

of correspondingly-indexed elements of U' and U''. One can then verify without difficulty that the elements of U are continuous and concave on the domain G (which is a nonnegative orthant in its own right), so that (N, G, A, U) is a market. Finally, one can verify without difficulty that (N, G, A, U) generates the game $(N; v'+v'')$. Q.E.D.

3.2 THE CORE THEOREM

THEOREM 4. *Every market game has a core.*

This theorem is well known, and has been generalized well beyond the limited class of markets we are now considering. Nevertheless we shall give two proofs, both short, for the sake of the insights they provide. In the first, we in effect determine a competitive equilibrium for the generating market (a simple matter when there is transferable utility), and then show that the competitive payoff vector lies in the core. In the second proof, we show directly that the game is balanced, and then apply Theorem 1.

Proof 1. Let $(N; v)$ be a market game and let (N, G, A, U) be a market that generates it. Let $B = \{b^i : i \in N\}$ be a feasible N-allocation that achieves the value $v(N)$ in (3-1) for $S = N$. The maximization in (3-1) ensures the existence of a vector p [competitive prices—but possibly negative!] such that for each $i \in N$, the expression

$$u^i(x^i) - p \cdot (x^i - a^i), \quad x^i \in G, \tag{3-4}$$

is maximized at $x^i = b^i$. Define the payoff vector β by

$$\beta^i = u^i(b^i) - p \cdot (b^i - a^i);$$

we assert that β is in the core. Indeed, let S be any nonempty subset of N, and let Y^S be a feasible S-allocation that achieves the maximum in (3-1), so that $v(S) = \sum_S u^i(y^i)$. Since b^i maximizes (3-4), we have

$$\beta^i \geqq u^i(y^i) - p \cdot (y^i - a^i).$$

Summing over $i \in S$, we obtain

$$\beta(S) \geqq \textstyle\sum_S u^i(y^i) - p \cdot 0 = v(S),$$

as required by (2-1). Moreover, if $S = N$ we may take $Y^S = B$ and obtain $\beta(N) = v(N)$, as required by (2-2). Q.E.D.

Proof 2. Let (N, G, A, U) be a generating market for $(N; v)$, and, for each $S \subseteq N$, let $Y^S = \{y_S^i: i \in S\}$ be a maximizing S-allocation in (3-1). Let $\mathscr{B}$ be balanced, with weights $\{\gamma_S^i: S \in \mathscr{B}\}$. Then we have

$$\sum_{S \in \mathscr{B}} \gamma_S v(S) = \sum_{S \in \mathscr{B}} \sum_{i \in S} \gamma_S u^i(y_S^i) = \sum_{i \in N} \sum_{\substack{S \in \mathscr{B} \\ S \ni i}} \gamma_S u^i(y_S^i).$$

Now define

$$z^i = \sum_{\substack{S \in \mathscr{B} \\ S \ni i}} \gamma_S y_S^i \in G, \quad \text{all } i \in N.$$

Note that z^i is a center of gravity of the points y_S, by virtue of (2-3). Hence, by concavity,

$$\sum_{S \in \mathscr{B}} \gamma_S v(S) \leqq \sum_{i \in N} u^i(z^i). \tag{3-5}$$

But $Z = \{z^i: i \in N\}$ is a feasible N-allocation, since

$$\sum_{i \in N} z^i = \sum_{S \in \mathscr{B}} \sum_{i \in S} \gamma_S y_S^i = \sum_{S \in \mathscr{B}} \gamma_S \sum_{i \in S} a^i = \sum_{i \in N} a^i.$$

Hence the right side of (3-5) is $\leqq v(N)$, and we conclude from (2-4) that the game is balanced and from Theorem 1 that it has a core. Q.E.D.

COROLLARY. *Every market game is totally balanced.*

Proof. If $(N; v)$ is generated by the market (N, G, A, U), and if $0 \subset R \subseteq N$, then we may define a market (R, G, A', U'), where A' and U' come from A and U by simply omitting all a^i and u^i for i not in R. This market clearly generates the game $(R; v)$. Hence $(R; v)$ is balanced.

Q.E.D.

Our next objective will be to prove the converse of this corollary—i.e. that every totally balanced game is a market game.

4. DIRECT MARKETS

A special class of markets, called *direct markets*, will play an important role in the sequel. They have the form

$$(T, E_+^T, I^T, \{u\}),$$

where u is homogeneous of degree 1 as well as concave and continuous. Here E_+^T denotes the nonnegative orthant of the vector space E^T with coordinates indexed by the members of T, and I^T denotes the collection of unit vectors of E^T—in effect, the identity matrix on T.

Thus, in a direct market, each trader starts with one unit of a personal commodity [e.g. his time, his labor, his participation, "himself"]. When it is brought together with other personal commodities, we may imagine that some desirable state of affairs is created, having a total value to the traders that is independent (because of homogeneity and equal tastes) of how they distribute the benefits.

Let e^S denote the vector in E^N in which $e_i^S = 1$ or 0 according as $i \in S$ or $i \notin S$; geometrically, these vectors represent the vertices of the unit cube in E_+^N. Then the characteristic function of the market game generated by a direct market can be put into a very simple form:

$$v(S) = u(e^S), \quad \text{all } S \subseteq N \tag{4-1}$$

(compare (3-3)). Note that only finitely many commodity bundles are involved in this expression.

4.1 The Direct Market Generated by a Game

Thus far we used markets to generate games. We now go the reverse route, associating with any game (not necessarily a market game) a certain "market of coalitions". Specifically, we shall say that the game $(N; v)$ "generates" the direct market $(N, E_+^N, I^N, \{u\})$, with u given by

$$u(x) = \max_{\{\gamma_S\}} \sum_{S \subseteq N} \gamma_S v(S), \quad \text{all } x \in E_+^N, \tag{4-2}$$

maximized over all sets of nonnegative γ_S satisfying

$$\sum_{S \ni i} \gamma_S = x_i, \quad \text{all } i \in N. \tag{4-3}$$

To explain this market,[5] we may imagine that each coalition S has an activity $\mathscr{A}_S$ that can earn $v(S)$ dollars if all the members of S participate fully. More generally, it earns $\gamma_S v(S)$ dollars if each member of S devotes the fraction γ_S of "himself" to $\mathscr{A}_S$. The maximization in (4-2) is then nothing but an optimal assignment of activity levels γ_S to the various $\mathscr{A}_S$'s, subject to the condition (4-3) that each player, i, distribute exactly the amount x_i of "himself" among his activities, including of course the "solo" activity $\mathscr{A}_{\{i\}}$.

The utility function defined by (4-2) is obviously homogeneous of degree 1, as required for a direct market. But before we can claim to have defined a market, let alone a direct market, we must also establish that (4-2) is continuous and concave. Continuity gives no trouble. To show concavity, it suffices (with homogeneity) to prove that

$$u(x) + u(y) \leqq u(x+y), \quad \text{all } x, y \in E_+^N.$$

[5] The essence of this model was suggested by D. Cantor and M. Maschler (private correspondence, 1962).

This is not difficult. By definition, there exist sets of nonnegative coefficients $\{\gamma_S\}$ and $\{\delta_S\}$ such that

$$u(x) = \sum_{S \subseteq N} \gamma_S v(S), \qquad u(y) = \sum_{S \subseteq N} \delta_S v(S);$$

and

$$\sum_{S \ni i} \gamma_S = x_i, \qquad \sum_{S \ni i} \delta_S = y_i, \qquad \text{all } i \in N.$$

Hence $\{\gamma_S + \delta_S\}$ is admissible for $x+y$, and (4-2) yields

$$u(x+y) \geqq \sum (\gamma_S + \delta_S) v(S) = u(x) + u(y),$$

as required.

4.2 The Cover of a Game

We shall now use the direct market generated by a game $(N; v)$ to generate in turn a new game $(N; \bar{v})$—schematically:

arbitrary game ⟶ direct market ⟶ market game.

We shall call $(N; \bar{v})$ the *cover* of $(N; v)$.

Combining (4-1) with (4-2) and (4-3), we obtain the following relation beteen v and $\bar{v}$:

$$\bar{v}(R) = \max_{\{\gamma_S\}} \sum_{S \subseteq R} \gamma_S v(S), \quad \text{all } R \subseteq N \tag{4-4}$$

maximized over $\gamma_S \geqq 0$ such that

$$\sum_{\substack{S \subseteq R \\ S \ni i}} \gamma_S = 1, \quad \text{all } i \in R. \tag{4-5}$$

Note that we could have taken (4-4), (4-5) as the definition of "cover", bypassing the intermediate market. Indeed, the cover of a game proves to be a useful mathematical concept quite apart from the present economic application.

We see immediately that

$$\bar{v}(R) \geqq v(R), \quad \text{all } R \subseteq N, \tag{4-6}$$

since one of the admissible choices for $\{\gamma_S\}$ in (4-4) is to take $\gamma_R = 1$ and all other $\gamma_S = 0$. Moreover, the equality cannot always hold in (4-6); indeed, $\bar{v}$ comes from a market game while v was arbitrary. Thus, the mapping $v \to \bar{v}$ takes an arbitrary characteristic function and, by perhaps increasing some values, turns it into the characteristic function of a market game.

Lemma 1. *If* $(N; v)$ *has a core, then* $\bar{v}(N) = v(N)$, *and conversely.*

Proof. Let α be in the core of $(N; v)$. Then

$$\begin{aligned}\bar{v}(N) &= \max_{\{\gamma_S\}} \sum_{S \subseteq N} \gamma_S v(S) \\ &\leqq \max_{\{\gamma_S\}} \sum_{S \subseteq N} \gamma_S \alpha(S) = \max_{\{\gamma_S\}} \sum_{i \in N} \alpha^i \sum_{S \ni i} \gamma_S \\ &= \max_{\{\gamma_S\}} \sum_{i \in N} \alpha^i = \alpha(N) \\ &= v(N),\end{aligned}$$

the successive lines being justified by (4-4), (2-1), (4-5), and (2-2). In view of (4-6) we therefore have $\bar{v}(N) = v(N)$.

Conversely, if $(N; v)$ has no core, then (2-4) fails for some balanced set $\mathscr{B}$ with weights $\{\gamma_S\}$. Defining $\gamma_S = 0$ for $S \notin \mathscr{B}$, we see that (4-5) holds (for $R = N$). Then (4-4) and the denial of (2-4) give us

$$\bar{v}(N) \geqq \sum_{S \subseteq N} \gamma_S v(S) = \sum_{S \in \mathscr{B}} \gamma_S v(S) > v(N);$$

Hence $\bar{v}(N) \neq v(N)$. Q.E.D.

LEMMA 2. *A totally balanced game is equal to its cover.*

Proof. Let $(N; \bar{v})$ be the cover of $(N; v)$, and let $O \subset R \subseteq N$. Then it is clear from the definitions that the cover of $(R; v)$ is $(R; \bar{v})$. But if $(N; v)$ is totally balanced, then $(R; v)$ has a core and $\bar{v}(R) = v(R)$ by Lemma 1. Hence $\bar{v} = v$. Q.E.D.

THEOREM 5. *A game is a market game if and only if it is totally balanced.*

Proof. We proved earlier (corollary to Theorem 4) that market games are totally balanced. We have just now shown that totally balanced games are equal to their covers, which are market games. Q.E.D.

4.3 EQUIVALENCE OF MARKETS

There is one more result of some heuristic interest that we can extract from the present discussion, before entering the realm of solution theory. This time, we follow the scheme:

arbitrary market $\longrightarrow$ market game $\longrightarrow$ direct market.

Let us call two markets *game-theoretically equivalent* if they generate the same market game. Then the two markets in the above scheme are equivalent in this way, since the cover of the game in the middle is just the market game of the market on the right, and these two games are equal by Lemma 2. This proves

THEOREM 6. *Every market is game-theoretically equivalent to a direct market.*

5. SOLUTIONS

An *imputation* for a game $(N; v)$ is a payoff vector α that satisfies

$$\alpha(N) = v(N) \tag{5-1}$$

and

$$\alpha^i \geqq v(\{i\}), \quad \text{all } i \in N. \tag{5-2}$$

A comparison with (2-1) and (2-2) shows that the imputation set is certainly not empty if the game has a core.[6]

Classical solution theory [23] rests upon a relation of "domination" between imputations. If α and β are imputations for $(N; v)$, then α is said to *dominate* β (written $\alpha \in\!\dashv \beta$) if there is some nonempty subset S of N such that

$$\alpha^i > \beta^i, \quad \text{all } i \in S, \tag{5-3}$$

and

$$\alpha(S) \leqq v(S). \tag{5-4}$$

A *solution* of $(N; v)$ is defined to be any set of imputations, mutually undominating, that collectively dominate all other imputations. Our only concern with this definition, technically, is to observe that it depends only on the concepts of imputation and domination; any further information conveyed by the characteristic function is disregarded.

The core is also closely dependent on these concepts. In fact, the core, when it exists, is precisely the set of undominated imputations. The converse is not universally true—there are some games that have undominated imputations but no core.[7] We can rule this out, however, by imposing the very weak condition:

$$v(S) + \sum_{N-S} v(\{i\}) \leqq v(N), \quad \text{all } S \subseteq N, \tag{5-5}$$

which is satisfied by all games likely to be met in practice.[8]

5.1 Domination-Equivalence

Two games will be called *d-equivalent* (domination-equivalent) if they have the same imputation sets and the same domination relations on them. It follows that *d*-equivalent games have precisely the same solutions, or lack of solutions. Also, if they have cores, they have the same cores;

[6] Some approaches to solution theory omit (5-2), relying on the solution concept itself to impose whatever "individual rationality" the situation may demand [5, 12]. This modification in the definition of solution would make little difference to our present discussion, except for eliminating the fussy condition (5-5). In particular, Theorem 7 and all of Section 6 would remain correct as written.

[7] We are indebted to Mr. E. Kohlberg for this observation.

[8] Thus, (5-5) is implied by either superadditivity or balancedness, but is weaker than both. For a game in normalized form, i.e. with $v(\{i\}) \equiv 0$, it merely states that no coalition is worth more than N.

moreover, within the class of games satisfying (5-5) the property of being balanced is preserved under d-equivalence. However, the property of being totally balanced is not so preserved, as the following lemma reveals.

LEMMA 3. *Every balanced game is d-equivalent to its cover.*

Proof. Let $(N; v)$ be balanced. By Lemma 1, $\bar{v}(N) = v(N)$ and by (4-4), $\bar{v}(\{i\}) = v(\{i\})$; hence the two games have the same imputations. Denote the respective domination relations by $\in\!\dashv$ and $\in\!\dashv'$. By (4-6) and (5-4) we see at once that the latter is, if anything, stronger than the former—i.e. $\alpha \in\!\dashv \beta$ implies $\alpha \in\!\dashv' \beta$. It remains to prove the converse.

Assume, *per contra*, that α and β are imputations satisfying $\alpha \in\!\dashv' \beta$ but not $\alpha \in\!\dashv \beta$. Then for some nonempty subset R of N we have

$$\alpha^i > \beta^i, \quad \text{all } i \in R,$$

and

$$\alpha(R) \leqq \bar{v}(R). \tag{5-6}$$

To avoid $\alpha \in\!\dashv \beta$ we must have

$$\alpha(S) > v(S) \tag{5-7}$$

for *all* S, $O \subset S \subseteq R$. Referring to the definition of $\bar{v}$, we see that there are nonnegative weights γ_S, $S \subseteq R$, such that

$$\bar{v}(R) = \sum_{S \subseteq R} \gamma_S v(S)$$

and

$$\sum_{\substack{S \subseteq R \\ S \ni i}} \gamma_S = i, \quad \text{all } i \in R.$$

Hence, using (5-7)

$$\bar{v}(R) < \sum_{S \subseteq R} \gamma_S \alpha(S) = \alpha(R).$$

The strict inequality here contradicts (5-6). Q.E.D.

By a "solution" of a market, we shall mean a solution of the associated market game.

THEOREM 7. *If $(N; v)$ is any balanced game whatever, then there is a market that has precisely the same solutions as $(N; v)$.*

Proof. The main work has been done in Lemma 3. Indeed, let $(N, E_+^T, I^T, \{u\})$ be the direct market generated by $(N; v)$. Then the solutions of this market are the solutions of $(N; \bar{v})$, which by the lemma is d-equivalent to $(N; v)$ and hence has the same solutions. Q.E.D.

5.2 A TECHNICAL REMARK

The notion of d-equivalence is essentially due to Gillies [*4*, *5*], though he works with a broader definition of imputation, not tied to the characteristic function by (5-1). He defines a *vital* coalition as one that achieves some domination that no other coalition can achieve, and shows that two games are d-equivalent (in the present sense) if and only if they have (i) the same imputation sets, (ii) the same vital coalitions, and (iii) the same v-values on their vital coalitions.

A necessary (but not sufficient) condition for a coalition to be vital is that it cannot be partitioned into proper subsets, the sum of whose v-values equals or exceeds its own v-value. Sufficiency would require the generalized partitioning provided by balanced sets.

Given a game $(N; v)$, we can define its "least superadditive majorant" $(N; \tilde{v})$ by

$$\tilde{v}(S) = \max \textstyle\sum_h v(S_h), \tag{5-8}$$

he maximization running over all partitions $\{S_h\}$ of S. (Compare (4-4), 4-5).) It can be shown that $\tilde{v}(N) = v(N)$ if and only if $(N; v)$ has a core (cf. Lemma 1 above), in which case the two games are d-equivalent. Thus, every game with a core is d-equivalent to a superadditive game.

However, as Gillies observes, d-equivalence can also hold nontrivially among superadditive games. That is, it may be possible to push the v-value of some nonvital coalition *higher* than the value demanded by superadditivity, without making the coalition vital.[9] We are using the full power of this observation, since the cover $\bar{v}$ can be thought of as the "greatest d-equivalent majorant" of v. Thus, $v \leqq \tilde{v} \leqq \bar{v}$, and all three may be different.

6. EXAMPLES

Lucas's 10-person game [8, 9] with no solution has players $N = \overline{1234567890}$ and the following characteristic function:

$$\left.\begin{aligned}
v(\overline{12}) = v(\overline{34}) = v(\overline{56}) = v(\overline{78}) = v(\overline{90}) &= 1 \\
v(\overline{137}) = v(\overline{139}) = v(\overline{157}) = v(\overline{159}) = v(\overline{357}) = v(\overline{359}) &= 2 \\
v(\overline{1479}) = v(\overline{2579}) = v(\overline{3679}) &= 2 \\
v(\overline{1379}) = v(\overline{1579}) = v(\overline{3579}) &= 3 \\
v(\overline{13579}) &= 4 \\
v(N) &= 5, \text{ and} \\
v(S) &= 0, \text{ all other } S \subseteq N.
\end{aligned}\right\} \tag{6-1}$$

The game has a core, containing among others the imputation that gives each "odd" player 1.[10] It is not superadditive (for example, $v(\overline{12})+v(\overline{34}) > v(\overline{1234})$); however it is d-equivalent to its least superadditive majorant $(N, \tilde{v})$, which can be calculated without difficulty,

[9] For example, at the end of Sec. 2, $v(\overline{123})$ may be increased from 1 to 3/2 without making 123 vital.

[10] The full core is a five-dimensional polyhedron, having vertices $e^S : S = \overline{13579}, \overline{23579}, \overline{14579}, \overline{13679}, \overline{13589}$, and $\overline{13570}$.

using (5-8). Moreover, one can verify that the latter is totally balanced, i.e. that $\tilde{v} = \bar{v}$ in this case. Thus, $(N, \tilde{v})$, defined by (6-1) and (5-8), is a market game with no solution.

The corresponding *market* with no solution, provided by Theorem 7, has ten traders and ten commodities, plus money. The traders have identical continuous concave homogeneous utilities $u(x)$, which may be calculated by applying (4-2), (4-3) to (6-1). Note that positive weights γ_S need be considered only for the eighteen vital coalitions and the ten singletons.[11] Of course, this is not the only utility function that works, since only a finite set of its values are actually used (cf. (4-1)).

6.1 A Production Model

Perhaps the most straightforward economic realization of Lucas's game is in the form of a production economy. (Compare the "activity" description in Section 4.1.) The production possibilities are generated by 18 specific processes, which produce the same consumer good (at constant

TABLE 1

Inputs										Output
x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}
1	1									1
		1	1							1
				1	1					1
						1	1			1
								1	1	1
1		1				1				2
1		1						1		2
1				1		1				2
1				1				1		2
		1		1		1				2
		1		1				1		2
1			1			1		1		2
	1			1		1		1		2
		1			1	1		1		2
1		1				1		1		3
1				1		1		1		3
		1		1		1		1		3
1		1		1		1		1		4

[11] The singleton weights are needed as slack variables, because we used "$=$" in (4-3) instead of "$\leq$".

returns to scale) out of various combinations of the raw materials (see Table I). Each entrepreneur starts with one unit of the correspondingly indexed raw material. The utility is simply the consumer good: $u(x) \equiv x_{11}$; hence it is not necessary to postulate a separate money.

This type of construction is perfectly general: a production model can be set up in a similar fashion for any other game in characteristic function form, one activity being required for each vital coalition. The market game generated by such a model will be the cover of the original game, and will have the same core and solutions provided that the original game was balanced.

6.2 Other Examples

Lucas [*6*, *7*, *10*] gives several examples of games in which the solution is unique but does not coincide with the core. In [*10*] he also describes a symmetric 8-person game, very similar to the above 10-person game, that has an infinity of solutions but none that treats the symmetric players symmetrically. Shapley [*18*] describes a 20-person game, of the same general type, every one of whose many solutions consists of the core, which is a straight line, plus an infinity of mutually disjoint closed sets that intersect the core in a dense point-set of the first category. A common feature of all these "pathological" examples is the existence of a core; hence, by Theorem 7, they are d-equivalent to market games that have the same solution behavior.

We close with another "pathological" example, of an older vintage [*14*], which because of its simple form leads to a direct market with utilities that we can write down explicitly. The game has players $N = \overline{123\ldots n}$, with $n \geqq 4$, and its characteristic function is given by

$$\left.\begin{array}{c} v(N-\{1\}) = v(N-\{2\}) = v(N-\{3\}) = v(N) = 1 \\ v(S) = 0, \quad \text{all other coalitions } S. \end{array}\right\} \qquad (6\text{-}2)$$

Thus, to win anything requires the participation of a majority of $\overline{123}$, plus all of the "veto" players $4, \ldots, n$. The core is the set of all imputations α that satisfy $\alpha_1 = \alpha_2 = \alpha_3 = 0$. It is easily verified that the game is totally balanced: $v = \bar{v}$. There are many solutions; but the remarkable feature of the game is a certain subclass of solutions, as follows:

Let B_e denote the set of imputation α that satisfy $\alpha_1 = 0, \alpha_2 = \alpha_3 \geqq e > 0$. Thus, B_e is a $(n-3)$-dimensional closed convex subset of the imputation space. In [*14*] it was shown that one may start with any closed subset of B_e whatever, and extend it to a solution of the game by adding only imputations that are at least $e/2$ distant from B_e.[12] The arbitrary starting

[12] The metric used here is $\rho(\alpha, \beta) = \max_i |\alpha_i - \beta_i|$. Our present claim entails a slight change in the construction given in [*14*], which merely keeps the rest of the solution away from the arbitrary subset of B_e, rather than from B_e itself.

set remains a distinct, isolated portion of the full solution. For example, if $n = 4$ (the simplest case), an arbitrary closed set of points on a certain line can be used.

To determine the direct market of this game, we apply (4-2), (4-3) to (6-2) and obtain the utility function

$$u(x) = \max_{\{\gamma^i\}} (\gamma^1+\gamma^2+\gamma^3),$$

maximized subject to

$$\left.\begin{array}{c} \gamma^1 \geqq 0, \qquad \gamma^2 \geqq 0, \qquad \gamma^3 \geqq 0; \\ \gamma^2+\gamma^3 \leqq x_1, \qquad \gamma^1+\gamma^3 \leqq x_2, \qquad \gamma^1+\gamma^2 \leqq x_3; \\ \gamma^1+\gamma^2+\gamma^3 \leqq x_i, \qquad i = 4,\ldots,n; \end{array}\right\}$$

where γ^i abbreviates $\gamma_{N-\{i\}}$. This reduces to the closed form:

$$u(x) = \min \left[x_1+x_2, x_1+x_3, x_2+x_3, \frac{x_1+x_2+x_3}{2}, x_4,\ldots,x_n\right]. \quad (6\text{-}3)$$

We see that u is the envelope-from-below of $n+1$ very simple linear functions.

Thus, an n-trader n-commodity market having the solutions containing arbitrary components, as described above, is obtained by giving the ith trader one unit of the ith commodity, $i = 1,\ldots, n$, and assigning them all the utility function (6-3).

REFERENCES

1. Bondareva, O. N., Nekotorye primeneniia metodov linejnogo programmirovaniia k teorii kooperativnykh igr (Some applications of linear programming methods to the theory of cooperative games). *Problemy Kibernetiki* **10** (1963), 119–139.

2. Charnes, A. and Kortanek, K. O., On balanced sets, cores, and linear programming. *Cahiers du Centre d'Etudes de Recherche Opérationelle* **9** (1967), 32–43.

3. Debreu, G. and Scarf, H., A limit theorem on the core of an economy. *International Economic Review* **4** (1963), 235–246.

4. Gillies, D. B., Some theorems on n-person games. Ph.D. Thesis, Princeton University, June 1953.

5. Gillies, D. B., Solutions to general non-zero-sum games. *Ann. Math. Study* **40** (1959), 47–85.

6. Lucas, W. F., A counterexample in game theory. *Management Science* **13** (1967), 766–767.

7. Lucas, W. F., *Games with Unique Solutions which are Nonconvex.* The RAND Corporation, RM–5363–PR, May 1967.

8. Lucas, W. F., A game with no solution. *Bull. Am. Math. Soc.* **74** (1968), 237–239; also The RAND Corporation, RM–5518–PR, November 1967.

9. Lucas, W. F., *The Proof that a Game may not have a Solution.* The RAND Corporation, RM–5543–PR, January 1968 (to appear in *Trans. Am. Math. Soc.*).

10. Lucas, W. F., *On Solutions for n-Person Games.* The RAND Corporation, RM–5567–PR, January 1968.

11. SCARF, H., The core of an n-person game. *Econometrica* **35** (1967), 50–69.

12. SHAPLEY, L. S., *Notes on the n-Person Game*—III: *Some Variants of the von Neumann–Morgenstern Definition of Solution.* The RAND Corporation, RM–817, April 1952.

13. SHAPLEY, L. S., *Markets as Cooperative Games.* The RAND Corporation, P–629, March 1955.

14. SHAPLEY, L. S., A solution containing an arbitrary closed component. *Ann. Math. Study* **40** (1959), 87–93; also The RAND Corporation, RM–1005, December 1952, and P–888, July 1956.

15. SHAPLEY, L. S., The solutions of a symmetric market game. *Ann. Math. Study* **40** (1959), 145–162; also The RAND Corporation, P–1392, June 1958.

16. SHAPLEY, L. S., *Values of Large Games*—VII: *A General Exchange Economy with Money.* The RAND Corporation, RM–4248, December 1964.

17. SHAPLEY, L. S., On balanced sets and cores. *Naval Research Logistics Quarterly* **14** (1967), 453–560; also The RAND Corporation, RM–4601–PR, June 1965.

18. SHAPLEY, L. S., *Notes on n-Person Games*—VIII: *A Game with Infinitely "Flaky" Solutions.* The RAND Corporation, RM–5481–PR (to appear).

19. SHAPLEY, L. S. and SHUBIK, M., "Concepts and theories of pure competition." *Essays in Mathematical Economics: In Honor of Oskar Morgenstern,* Martin Shubik (ed.). Princeton University Press, Princeton, New Jersey, 1967, 63–79; also The RAND Corporation, RM–3553–PR, May 1963.

20. SHAPLEY, L. S. and SHUBIK, M., Quasi-cores in a monetary economy with non-convex preferences. *Econometrica* **34** (1966), 805–827; also The RAND Corporation, RM–3518–1, October 1965.

21. SHAPLEY, L. S. and SHUBIK, M., *Pure Competition, Coalitional Power, and Fair Division.* The RAND Corporation, RM–4917–1–PR, March 1967 (to appear in *International Economic Review*).

22. SHUBIK, M., Edgeworth market games. *Ann. Math. Study* **40** (1959), 267–278.

23. VON NEUMANN, J. and MORGENSTERN, O., *Theory of Games and Economic Behavior.* Princeton University Press, Princeton, New Jersey, 1944, 1947, 1953.

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