Available online at www.sciencedirect.com
ScienceDirect
Games and Economic Behavior 62 (2008) 329–347
GAMES and Economic Behavior
www.elsevier.com/locate/geb

# Multiperson utility

Manel Baucells [a,*], Lloyd S. Shapley [b]

[a] Department of Managerial Decision Sciences, IESE Business School, Ave. Pearson 21, 08034 Barcelona, Spain
[b] Departments of Economics and Mathematics, University of California at Los Angeles, Los Angeles, CA 90095-1477, USA

Received 25 May 2006
Available online 30 August 2007

**Abstract**

We approach the problem of preference aggregation by endowing both individuals and coalitions with partially-ordered or incomplete preferences for decision under risk. Restricting attention to the case of complete individual preferences, and assuming complete preferences for *some* pairs of agents (interpersonal comparisons of utility units), we discover that the Extended Pareto Rule (if two disjoint coalitions $A$ and $B$ prefer $x$ to $y$, then so does the coalition $A \cup B$) imposes a "no arbitrage" condition in the terms of utility comparison between agents. Furthermore, if all the individuals and pairs have complete preferences and certain non-degeneracy conditions are met, then we witness the emergence of a complete preference ordering for coalitions of all sizes. The corresponding utilities are a weighted sum of individual utilities, with the $n-1$ independent weights obtained from the preferences of $n-1$ pairs forming a spanning tree in the group.


*JEL classification:* D60; D70; D71; D81

*Keywords:* Preference aggregation; Incomplete preferences; Extended Pareto Rule

## 1. Introduction

In the modeling of social or group preferences, it is common to assume that groups have transitive and complete preference orderings (Harsanyi, 1955; Arrow, 1963; Karni, 2003; Dhillon and Mertens, 1999).[1] We feel that such assumption has been taken for granted, and it deserves

* Corresponding author. Fax: +34 011 3493 253 4343.
*E-mail addresses:* mbaucells@iese.edu (M. Baucells), shapley@math.ucla.edu (L.S. Shapley).

[1] An exception here is Sen's (1970) study of incomplete group preferences.

0899-8256/$ – see front matter 
doi:10.1016/j.geb.2007.07.002

more attention and discussion because groups should not be conceived to act as individuals from the outset.

Our approach to the problem of modeling group preferences begins by endowing coalitions with cardinal preference orderings that may fail to be complete, i.e., some pairs of outcomes may be regarded as incomparable. Allowing incompleteness in the group preference is particularly appealing because one should not expect that the group is able to agree on how to compare every pair of alternatives. Rather, if we insist on a unanimous agreement, a group will very likely produce only a partial ordering of the alternatives.

Given this relaxation, our aim is to provide a minimal set of sufficient conditions that lead to complete preferences for the group and the rest of coalitions, thus giving an axiomatic basis for the existence of a cardinal social utility function which is additive in individual utilities. Even at an individual level, the assumption of completeness is not innocuous, as argued by Von-Neumann and Morgenstern (1944). In fact, Aumann's (1962) groundbreaking work on incomplete preferences, as well as the more recent work by Dubra et al. (2004), deals with individuals. Here, we will assume individuals exhibit completeness, but the coalitions do not.[2]

Although Von-Neumann and Morgenstern (1944), in an appendix of the 2nd edition, introduced axioms to justify the use of cardinal utility as required by their mixed strategies, they failed to provide similar axiomatic support for the use of comparable and transferable utilities[3] in the characteristic-function games that occupy the final two thirds of their great book. The present work is a step in a long-term project, first envisioned by the senior author more than 40 years ago, to fill this gap with the aid of incomplete preferences.[4]

We will present a few technical conditions, but given these, a weak property called the Extended Pareto Rule (*EPR*) is necessary and sufficient for all coalitions to have utility functions (the *EPR* says that if two disjoint coalitions $A$ and $B$ prefer $x$ to $y$, then their union prefers $x$ to $y$). That *EPR* may have important consequences for the social aggregation of individual preferences was already claimed in Shapley and Shubik (1982, p. 66):

**A** "*With the Pareto Principle thus strengthened, we can often weaken some of the other hypotheses [regarding completeness]... and still obtain the existence of a social utility function. For example, we can [assume completeness]... only for two-member groups. ... With the aid of the Extended Pareto Rule, we can then derive utility functions for all other subsets of $N$, including $N$ itself.*"
Moreover (p. 68), "*even stronger conclusions can sometimes be drawn when we are working with conditions that lead to cardinal utility.*"

In this paper we formalize and prove **A** utilizing a cardinal framework. In this regard, this paper can be seen as an extension of Harsanyi's (1955) work. The multiperson setup does not add or subtract any support to the normative appeal of the Independence Axiom that leads to cardinal or linear utility. Because linear utilities are a special case, it should be understood that our work is restricted to coalitions that abide by the independence axiom. Equipped with this

[2] Several studies such as Bewley (1986), Schmeidler (1969), and Rigotti and Shannon (2004) explore the implications of incomplete individual preferences.

[3] Of course, transferable utility is even more demanding than comparable utility because one needs to further assume that there is a kind of risk-linear "money," with which the transfers may be carried out. These transfers may be realized by exchanging this "money" commodity in accordance with the utility comparison rates that we derive.

[4] We know of no indication that either von Neumann or Morgenstern were ever aware of this logical gap.

axiom and the Pareto rule, Harsanyi showed that the utility representing the group preference is a weighted sum of individual utilities. We use the same framework, but relax the completeness axiom.

Section 2 begins with a brief review of the theory of incomplete preferences. Next, it presents our first important result, a characterization of *EPR* in an environment where all the coalitions are endowed with incomplete preferences for risky prospects. We progressively explore the implications of this characterization when individuals are assumed to have complete preferences. Next, we introduce bilateral agreements, or pair preferences that happen to be complete. The utility of such bilateral agreements takes an additive form. If individual $i$ has utility $u_{\bar{i}}$, the utility of the pair $\overline{ij}$ takes the form $u_{\overline{ij}} = (u_{\bar{i}} + \delta_{i,j} u_{\bar{j}})/(1 + \delta_{i,j})$.[5] The weight $\delta_{i,j}$ serves as a comparison rate of utility units (utility differences), that we call the *utility comparison rate* between $i$ and $j$.

With three (or more) individuals we find that *EPR* is surprisingly powerful in forcing completeness of group preferences. If the pairs $\overline{12}$ and $\overline{23}$ have complete preferences, with utility comparison rates $\delta_{1,2}$ and $\delta_{2,3}$, then the preference for the triple $\overline{123}$ is necessarily complete and is given by $u_{\overline{123}} = (u_{\bar{1}} + \delta_{1,2} u_{\bar{2}} + \delta_{1,2}\delta_{2,3} u_{\bar{3}})/(1 + \delta_{1,2} + \delta_{1,2}\delta_{2,3})$. Interestingly, we find that *EPR* implies a "no arbitrage" condition in the utility comparison rates, analogous to the relations that hold among currency exchange rates, namely, that if $\overline{13}$ were to have a complete preference, then it necessarily has the utility comparison rate $\delta_{1,3} \equiv \delta_{1,2}\delta_{2,3}$.

Section 3 presents the main result, which is the extension of the above findings to $n$ individuals. They are both stated under minimal requirements regarding linear independence of the individual utilities (*EPR* allows us to relax substantially the Independent Prospects condition implicit in Harsanyi's theorem). In essence, and barring degeneracy, if we are given complete preferences for *some* pairs forming a spanning tree in the complete graph where the individuals are the nodes, then *EPR* implies the emergence of a complete preference for some more coalitions, including the set $N$ of all individuals. It follows that if *all* the pairs have complete preferences, then *all* the coalitions have complete preferences. Any such complete preference is represented by a weighted sum of the individual utilities, i.e., $u_S = \sum_{i \in S} \lambda_i u_{\bar{i}}$ for all $S \subseteq N$. The $n-1$ weights are determined up to a positive multiple from the utility comparison rates between any $n-1$ bilateral agreements forming a spanning tree.

Harsanyi's formulation does not provide a clue on how to obtain these weights. Here, we provide some guidance, the weights can be uniquely obtained by means of bilateral agreements between $n-1$ pairs of individuals. Moreover, these $n-1$ pairs cannot be arbitrarily chosen, but must form a spanning tree in the graph where the players are the nodes. This insight might prove fruitful to develop practical procedures to reach group consensus (Baucells and Sarin, 2003).

Section 4 discusses the previous results. For example, the geometrical representation provided by the cardinal framework uncovers a surprising connection with Desargues' Theorem, a geometrical result attributed to the 17th century French mathematician Girard Desargues. We also point out the relation between "stability" of the group preference and a very mild assumption called "minimal consensus," namely, that there exist two prospects $x, y$ such that all individuals strictly prefer $x$ to $y$. In the absence of minimal consensus, the group preference might exhibit disproportionate sensitivity to the terms of utility comparison produced by any given bilateral agreement. We also introduce a certain weakening of *EPR* in the subsection "Masters and Servants." Section 5 concludes and suggests some extensions. Proofs are provided in Appendix A.

[5] The overhead bar indicates set membership. Thus, "$\overline{ij}$" is a concise synonym for "$\{i, j\}$".

## 2. Incomplete coalition preferences and the Extended Pareto Rule

### *2.1. Review of the theory of incomplete preferences*

The underlying domain of *prospects* over which the preferences are given is $\mathcal{M}$, a closed, convex subset of $\mathbb{R}^m$. We further stipulate $\mathcal{M}$ to have dimension $m$, so that $\mathcal{M}$ contains interior points. For example, the simplex $\mathcal{M} = \{(x_1, \ldots, x_m) \in \mathbb{R}^m\colon\ \sum_{k=1}^m x_k \leqslant 1, x_k \geqslant 0\}$ could represent the set of probability mixtures over $m + 1$ "pure" prospects $k \in \{0, 1, \ldots, m\}$, where the pure prospect $k = 0$ occupies the origin of $\mathbb{R}^m$. Probability mixtures of two prospects $x$ and $y$ are identified with the prospect $\alpha x + (1 - \alpha)y$, for suitable $\alpha \in [0, 1]$.

We stipulate the following four axioms for an *incomplete preference* relation $\gtrsim$ —they are asserted for all $x, y, z \in \mathcal{M}$ and all $\alpha \in [0, 1]$:

$(P1)$ *Reflexivity*: $x \gtrsim x$.

$(P2)$ *Transitivity*: If $x \gtrsim y$ and $y \gtrsim z$, then $x \gtrsim z$.

$(P3)$ *Independence*: For all $\alpha \neq 0$, $x \gtrsim y$ if and only if $\alpha x + (1 - \alpha)z \gtrsim \alpha y + (1 - \alpha)z$.

$(P4)$ *Continuity*: The set $\{\alpha : x \gtrsim \alpha y + (1 - \alpha)z\}$ is closed.

Besides the strict preference and indifference relations induced by $\gtrsim$ (defined in the usual way), it is also possible that neither $x \gtrsim y$ nor $y \gtrsim x$, i.e., that $x$ and $y$ are *incomparable*. If $\mathcal{M}$ has no incomparable pairs then $\gtrsim$ is said to be *complete*; this can be expressed axiomatically by replacing $(P1)$ with

$(P1')$ *Completeness*: Either $x \gtrsim y$ or $y \gtrsim x$.

Let us denote by $\mathcal{M}^*$ the set of all real-valued functions on $\mathcal{M}$ that are both *linear* and *homogeneous*, i.e., $u(\alpha x) = \alpha u(x)$. In our finite-dimensional setting, $\mathcal{M}^*$ coincides with $(\mathbb{R}^m)^*$, so that the space of all linear homogeneous functions on $\mathbb{R}^m$ can be viewed as a copy of $\mathbb{R}^m$. Thus, if $u \in \mathcal{M}^*$, then $u(x)$ becomes the *inner product* of the vector $u = (u^1, \ldots, u^m)$ and the prospect $x = (x_1, \ldots, x_m)$. If $x$ is a probability mixture, then $u(x)$ is the "expected utility" of $x$.

Profiting from the recent work of Dubra et al. (2004), the previous four axioms lead to the following representation theorem:

**Theorem 1.** (a) *If $\gtrsim$ is an incomplete preference relation defined on $\mathcal{M}$, then there exists a non-empty subset $U \subseteq \mathcal{M}^* = R^m$ such that for all $x, y \in \mathcal{M}$,*

$$x \gtrsim y \Leftrightarrow u(x) \geqslant u(y) \quad \textit{for all } u \in U. \tag{1}$$

*Conversely, given any set $U \subseteq \mathbb{R}^m$, the relation defined by* (1) *is an incomplete preference relation.*

The details and proofs of Theorem 1 in this general setting can be consulted in Dubra et al. (2004), as well as in the preliminary section of Shapley and Baucells (1998). See also Vind (2000) for related work in large spaces, as well as Nau's (2006) for extending a similar formulation to both utility and probability.

Without loss of generality, the set $U$ in (1) can be taken to be a closed convex cone in $\mathbb{R}^m$, not containing the origin except if $U = \{0\}$. This can always be done by replacing any given set $U$ by its double polar, $(U^*)^*$ (Rockafellar, 1970, p. 121). If $\gtrsim$ is a non-trivial complete preference, then $U$ is a half-ray point at, but not containing, the origin. Hence, any element of $U$ is a non-negative multiple of any other element of $U$. In this case, we pick any $u \in U$, different from 0 unless $U = \{0\}$, and say that $\gtrsim$ is complete and has utility $u$.

*2.2. Definition and representation of the Extended Pareto Rule*

Given a set $N = \{1, \ldots, n\}$ of individuals, we fix $\mathcal{M}$ as the common prospect space. We endow each non-empty *coalition* $S \subseteq N$ with an incomplete preference $\succsim_S$ on $\mathcal{M}$, and let $U_S$ be its corresponding utility cone. Unless stated otherwise, the term coalition refers to a non-empty coalition. We shall restrict our attention in this account to the case where singletons (i.e., the individuals) have non-trivial, complete preferences $\succsim_{\bar{i}}$ with utility $0 \neq u_{\bar{i}} \in \mathbb{R}^m$, for all $i \in N$. Of course, the utility cone $U_{\bar{i}}$ associated with $\succsim_{\bar{i}}$ is the ray of positive multiples of $u_{\bar{i}}$.

Let $Sp(u_1, u_2, \ldots, u_k)$ denote the vector subspace of $\mathbb{R}^m$ spanned by some collection of utilities, and $d(u_1, u_2, \ldots, u_k)$ its dimension.

One might want to identify the coalition preference $\succsim_S$ with the *Paretian preference* $\overset{p}{\succsim}_S$ given by the unanimity rule: for all $x, y \in \mathcal{M}$,

$$x \overset{p}{\succsim}_S y \Leftrightarrow x \succsim_{\bar{i}} y \quad \text{for all } i \in S. \tag{2}$$

$\overset{p}{\succsim}_S$ is an incomplete preference in the sense of Axioms $P1 - P4$, and has utility cone $U_S^p = \mathrm{Co}(\bigcup_{i \in S} U_{\bar{i}})$, where $\mathrm{Co}(\cdot)$ indicates the convex hull. Unless all the members of $S$ share identical preferences, the Paretian preference will contain incomparable pairs so that for *completeness* of group preference to arise we need the ability of certain coalitions to establish comparisons beyond the ones given in (2).

Our setting, which treats individuals as one-person coalitions, allows for an important strengthening of the Pareto rule. A collection of preferences $\succsim_S$, $S \subseteq N$, satisfies the *Extended Pareto Rule* (*EPR*) if for all *disjoint* coalitions $A$ and $B$, and for all $x, y \in \mathcal{M}$,[6]

$$x \succsim_A y, x \succsim_B y \Rightarrow x \succsim_{A \cup B} y, \quad \text{and} \tag{3}$$

$$x \succ_A y, x \succsim_B y \Rightarrow x \succ_{A \cup B} y. \tag{4}$$

As stated, *EPR* is equivalent to the seemingly more general rule in which the statements corresponding to (3) and (4) hold for any partition.[7] In particular, the two *EPR* conditions imply the corresponding Pareto conditions:

$$x \succsim_i y \quad \text{for all } i \in N \Rightarrow x \succsim_N y, \quad \text{and} \tag{5}$$

$$x \succsim_i y \quad \text{for all } i \in N, \quad \text{and} \quad x \succ_j y \quad \text{for some } j \in N \Rightarrow x \succ_N y. \tag{6}$$

Let $\mathrm{Co}_{A,B}$ denote the convex hull of $U_A \cup U_B$, and $\mathrm{Co}_{A,B}^{ri}$ denote the set of relative interior points in $\mathrm{Co}_{A,B}$.[8] *EPR* can be characterized in terms of utility cones as follows.

---

[6] *EPR* has been recently used in Dhillon (1998) and Dhillon and Mertens (1999). A similar condition is used in the literature on conditional preferences (Luce and Krantz, 1971; Fishburn, 1973; Skiadas, 1997), where it is assumed that a decision maker possesses a collection of preferences conditional on events. Such collection satisfies *EPR* as applied to disjoint events. Our results indicate that the completeness axiom is redundant in some of these conditional preferences, and sets a framework to study incomplete conditional preferences. This last application is related to Bewley's (1986) work on "Knightian uncertainty," where he argues that incompleteness is natural in an environment with uncertainty (randomness with unknown probability distributions).

[7] For example, if $\{A, B, C\}$ is a partition of $S$ and $x \succ_A y$, $x \succsim_B y$, and $x \succsim_C y$, then imposing (4) to $A$ and $B$ produces $x \succ_{A \cup B} y$; and imposing (4) to $A \cup B$ and $C$ yields $x \succ_S y$.

[8] In finite-dimensional spaces, a point is *relatively internal* to a convex set if and only if it is relatively interior (in the usual topology). Formally, given that $U_A$ and $U_B$ are convex sets, a point

$$u^* \in \mathrm{Co}_{A,B} \equiv \{(1-\alpha)u_A + \alpha u_B \in \mathbb{R}^m \colon u_A \in U_A, u_B \in U_B, 0 \leqslant \alpha \leqslant 1\}$$

**Theorem 2.** *The Extended Pareto Rule holds if and only if for any two disjoint coalitions A and B,*

$$U_{A\cup B} \subseteq \mathrm{Co}_{A,B}, \quad \textit{and} \tag{7}$$

$$U_{A\cup B} \cap \mathrm{Co}^{ri}_{A,B} \neq \emptyset. \tag{8}$$

*2.3. Harsanyi's theorem*

Because (3) and (4) imply (5) and (6), the same argument in Theorem 2 shows that the latter two conditions are equivalent to $U_N \subseteq \mathrm{Co}_N \equiv \mathrm{Co}(U_i, i \in N)$ and $U_N \cap \mathrm{Co}^{ri}_N \neq \emptyset$. If both the individuals and the whole group are assumed to have complete preferences, then both $U_i$, $i \in N$ and $U_N$ are utility rays represented by positive multiples of some utility functions $u_i$, $i \in N$ and $u_N$. That $u_N \in \mathrm{Co}_N$ and $u_N \cap \mathrm{Co}^{ri}_N \neq \emptyset$ imply that $u_N$ can be expressed as weighted sum of the $u_i$'s, and that these weights are strictly positive. In this setting, the weights are unique, up to a positive multiple, if $d(u_{\bar{1}}, u_{\bar{2}}, \ldots, u_{\bar{n}}) = n$. This is precisely Harsanyi's (1955) aggregation theorem, which is easily derived using our framework of utility cones.

In this paper we shall use the following special case of Theorem 2, which coincides with Harsanyi's Theorem for $n = 2$.

**Corollary 3.** *Let A and B be two disjoint coalitions, and for $S \in \{A, B, A \cup B\}$, assume that $\succsim_S$ is complete and has utility $u_S$. Then* (3) *and* (4) *hold if and only if $\succsim_{A\cup B}$ has utility $u_{A\cup B} = \lambda[(1-\alpha)u_A + \alpha u_B]$, for some $0 < \alpha < 1$ and $\lambda > 0$.*

*2.4. Bilateral agreements*

A complete preference for a pair $\overline{ij}$ is called a *bilateral agreement*. By setting $\lambda = 1$ in Corollary 3 we have that a bilateral agreement $\succsim_{\overline{ij}}$ has utility $u_{\overline{ij}} = (1-\alpha_{i,j})u_{\bar{i}} + \alpha_{i,j}u_{\bar{j}}$, for some $0 < \alpha_{i,j} < 1$. Letting $\delta_{i,j} \equiv \alpha_{i,j}/(1-\alpha_{i,j}) \in (0,\infty)$ we shall prefer to write

$$u_{\overline{ij}} = (u_{\bar{i}} + \delta_{i,j}u_{\bar{j}})/(1+\delta_{i,j}). \tag{9}$$

If $d(u_{\bar{i}}, u_{\bar{j}}) = 2$, i.e., $u_{\bar{i}}$ and $u_{\bar{j}}$ are linearly independent, then $\delta_{i,j}$ is unique. This single parameter has the natural interpretation of a "utility comparison rate" between $i$ and $j$. Note that "$\overline{ij}$" is not an ordered set, but the order of $i$ and $j$ in $\delta_{i,j}$ matters. In fact, $\delta_{j,i} = 1/\delta_{i,j}$ and $u_{\overline{ij}} = (\delta_{j,i}u_{\bar{i}} + u_{\bar{j}})/(1+\delta_{j,i})$.

The choice of $\delta_{i,j}$ is exogenous in our model. In fact, it can encompass the idea that different individuals may have different susceptibilities to satisfaction, as discussed by Harsanyi (1955, p. 318). Several methods to elicit the value of $\delta_{i,j}$ are discussed in Baucells and Sarin (2003). Because we can re-scale the individual utilities, the selection of $\delta_{i,j} = 1$ should not be associated with a "fair" or symmetric pair agreement (Dhillon and Mertens, 1999; Karni, 2003; Sobel, 2001).

Of course, the assumption of a complete pair agreement, with its representation by a single utility comparison rate, is a strong assumption. Our framework allows for a natural relaxation of this assumption in which an incomplete pair agreement $\succcurlyeq_{\overline{ij}}$ is represented by an interval $[\delta^{\ell}_{i,j}, \delta^{h}_{i,j}]$ of utility comparison rates. If the pair just accepts unanimity as the basis to form

is relatively internal to $\mathrm{Co}_{A,B}$ if for all $u \in \mathrm{Co}_{A,B}$ there is a $u'$ such that $u^* = (1-\alpha)u + \alpha u'$ for some $0 < \alpha < 1$.

pair preferences, then $\delta^{\ell}_{i,j} = 0$ and $\delta^{h}_{i,j} = \infty$. In general, one would hope for an intermediate resolution where $0 < \delta^{\ell}_{i,j} \leqslant \delta^{h}_{i,j} < \infty$.

### *2.5. Three individuals and "no arbitrage" in utility comparison rates*

To visualize the restrictions that *EPR* imposes on preferences consider a special case with $m = 3$, which would be the case if there are just four pure prospects, and $n = 3$. Assume $d(u_{\bar{1}}, u_{\bar{2}}, u_{\bar{3}}) = 3$ and let $W$ be the plane in $\mathbb{R}^3$ containing the points $u_{\bar{1}}$, $u_{\bar{2}}$, and $u_{\bar{3}}$. These points can be pictured as the intersection of $W$ with the rays $U_{\bar{i}}$, with the origin $(0, 0, 0)$ not in $W$. For $i \neq j$, if $\succsim_{\overline{ij}}$ is some incomplete preference, then the intersection of the utility cone $U_{\overline{ij}}$ with $W$ is a closed line segment contained in $u_{\bar{i}}u_{\bar{j}}$ (the line segment between $u_{\bar{i}}$ and $u_{\bar{j}}$). In Fig. 1 we abuse notation and use $U_S$ to indicate such intersections.

By applying *EPR* to all the partitions of $\overline{123}$ we obtain

$$U_{\overline{123}} \subseteq \hat{U}_{\overline{123}} \equiv \mathrm{Co}(U_{\bar{1}} \cup U_{\overline{23}}) \cap \mathrm{Co}(U_{\bar{2}} \cup U_{\overline{13}}) \cap \mathrm{Co}(U_{\bar{3}} \cup U_{\overline{12}}). \tag{10}$$

If the pair preferences coincide with the Paretian preference, then $U^{p}_{\overline{ij}} \cap W$ is just the closed line segment $u_{\bar{i}}u_{\bar{j}}$ and (10) does not restrict $U_{\overline{123}} \cap W$. However, if the pair preferences are more nearly complete, then $U_{\overline{ij}} \cap W$ is strictly contained in $u_{\bar{i}}u_{\bar{j}}$ and (10) begins to be very effective in restricting $\hat{U}_{\overline{123}}$, and hence $\gtrsim_{\overline{123}}$. In particular, if $\hat{U}_{\overline{123}} \cap W$ were a point, then $\gtrsim_{\overline{123}}$ would necessarily be complete. But notice that $\gtrsim_{\overline{123}}$ *has to be* complete if $\gtrsim_{\overline{12}}$ and $\gtrsim_{\overline{23}}$ are complete. Applying (10), the utility rays associated with $u_{\overline{23}}$ and $u_{\overline{12}}$, along with $u_{\bar{1}}$, $u_{\bar{3}}$, determine a unique utility ray $U_{\overline{123}}$ represented by $u_{\overline{123}}$, the intersection of $u_{\bar{1}}u_{\overline{23}}$ and $u_{\overline{12}}u_{\bar{3}}$. Thus, a complete preference $\gtrsim_{\overline{123}}$ emerges from two bilateral agreements.

Figure 2 illustrates this fact, and it also reveals that $U_{\overline{13}}$ has to include the utility labeled as $u_{\overline{13}}$, the intersection of the line segment $u_{\bar{1}}u_{\bar{3}}$ and the extension of $u_{\bar{2}}u_{\overline{123}}$; otherwise $U_{\overline{123}} \subseteq \mathrm{Co}(U_{\bar{2}} \cup U_{\overline{13}})$ fails. If $\gtrsim_{\overline{13}}$ were a complete preference, then $u_{\overline{13}} \in U_{\overline{13}}$ would be the unique utility candidate consistent with *EPR*.

If the bilateral agreement between 1 and 3 were not to contain $u_{\overline{13}}$, then *EPR* would not hold. Specifically, one could find prospects $x$ and $y$ such that $x \succsim_{\bar{2}} y$ and $x \succsim_{\overline{13}} y$, but the coalition $\overline{123}$ would not weakly prefer $x$ over $y$. This generalizes the violation of the Pareto rule that occurs if $U_{\overline{12}}$ were not contained in $\mathrm{Co}(U_{\bar{1}} \cup U_{\bar{2}})$.

To illustrate with an example this violation, suppose I am a junior professor who is "bad" at negotiating with superiors, but "good" at negotiating with equals. In our terminology, this means that the utility of a pair containing me and any one of my superiors is close to my superior's

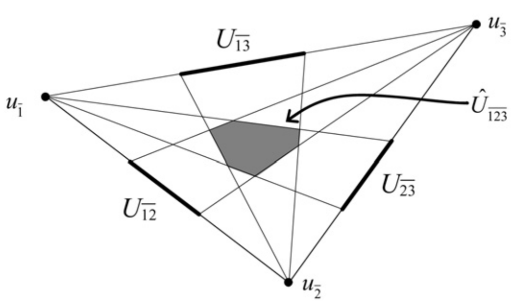


Fig. 1. Geometrical illustration of *EPR*. $U_{\overline{123}}$ must lie within $\hat{U}_{\overline{123}}$.

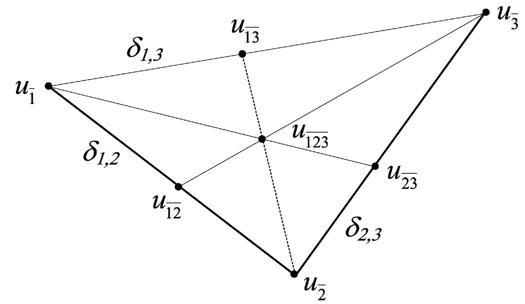


Fig. 2. Given $u_{\overline{12}}$ and $u_{\overline{23}}$, $\succsim_{\overline{123}}$ is complete and $\delta_{1,3} = \delta_{1,2}\delta_{2,3}$.

utility; and the utility of a pair in which there is me and another junior is close to my utility. We also suppose that everybody else is such that the utility of a two person coalition is the average utility. Then, the arbitrage condition does not hold, and neither does *EPR*.

The location of $u_{\overline{13}}$ in the geometrical construction of Fig. 2 has an interesting "no arbitrage" property. Let $\delta_{1,2}$ and $\delta_{2,3}$ be the utility comparison rates of the bilateral agreements $\succsim_{\overline{12}}$ and $\succsim_{\overline{23}}$, respectively. If $\delta_{1,2}$ "utils" of individual 1 are comparable to one "util" of individual 2; and $\delta_{2,3}$ "utils" of individual 2 are comparable to one "util" of individual 3, then it should the case that $\delta_{1,2}\delta_{2,3}$ "utils" of individual 1 are comparable to one "util" of individual 3. Similar to "no arbitrage" in currency exchange rates, the natural utility comparison rate between 1 and 3 is $\delta_{1,3} = \delta_{1,2}\delta_{2,3}$. This indeed holds for the $\delta_{1,3}$ associated with $u_{\overline{13}}$, since this equality is well known, mostly in its symmetric form $\delta_{1,2}\delta_{2,3}\delta_{3,1} = 1$, as Ceva's Theorem (Brannan et al., 1999, pp. 75–76).

Lemma 7 in Appendix A formalizes the results of this sub-section. In summary, we begin with bilateral agreements $\gtrsim_{\overline{12}}$ and $\gtrsim_{\overline{23}}$ with utility comparison rates $\delta_{1,2}$ and $\delta_{2,3}$ that determine $u_{\overline{12}}$ and $u_{\overline{23}}$. Next, we let $\delta_{1,3} \equiv \delta_{1,2}\delta_{2,3}$. Then, under some non-degeneracy conditions, $\gtrsim_{\overline{123}}$ is complete and has utility $u_{\overline{123}} = (u_{\bar{1}} + \delta_{1,2}u_{\bar{2}} + \delta_{1,3}u_{\bar{3}})/(1 + \delta_{1,2} + \delta_{1,3})$. Moreover, if $u_{\bar{2}} \notin Sp(u_{\bar{1}}, u_{\bar{3}})$, then $u_{\overline{13}} \in U_{\overline{13}}$, so that if $\gtrsim_{\overline{13}}$ is complete, then it necessarily has utility $u_{\overline{13}} = (u_{\bar{1}} + \delta_{1,3}u_{\bar{3}})/(1 + \delta_{1,3})$.

## 3. The case of *n* individuals: a utility comparison system

To generalize the previous construction to more than three individuals requires at least one comparison channel between each pair of individuals. If we view the individuals as the nodes of a graph and the bilateral agreements as the edges, then this requirement means a connected graph. The "no arbitrage" condition indicates that a chain of bilateral agreements that "cycles" (starts and finishes in the same individual) contains redundancies. Therefore, if individuals are seen as nodes and bilateral agreements as edges of a graph, what is needed is a set of edges that form a connected and acyclic graph, i.e., a spanning tree. We now proceed to show that, for a given spanning tree of bilateral agreements, the preferences of certain "connected" coalitions are complete.

To formalize this, we begin by introducing some definitions. An (undirected) graph is pair $(N, \mathcal{G})$, where $\mathcal{G}$ is a collection of two-member coalitions of $N$. If $\overline{ij} \in \mathcal{G}$, then we say that $i$ is *adjacent* to $j$ in $(N, \mathcal{G})$. Individual $i$ is *connected* to individual $j \neq i$ in $(N, \mathcal{G})$ if there is a sequence of individuals $(i = i_1, i_2, \ldots, i_k = j)$ in $N$ such that $\overline{i_r i_{r+1}} \in \mathcal{G}$ for every $r \in \{1, \ldots, k-1\}$. Any such sequence is called a *path* in $(N, \mathcal{G})$; it is a *simple* path iff no individual is repeated. $\mathcal{T}$ is a *spanning tree* of $N$ iff there is a unique simple path in $(N, \mathcal{T})$ connecting any individual to any other individual. It follows that $\mathcal{T}$ contains precisely $n-1$ pairs. We use $\mathcal{T}$ instead of $\mathcal{G}$ whenever $\mathcal{T}$ is a spanning tree of $N$.

Equipped with a spanning tree $\mathcal{T}$, and the respective utility comparison rates between ordered pairs in $\mathcal{T}$, we propose the appropriate weights to determine the utilities for coalitions. We chose an arbitrary "*numeraire*" individual, say $i = 1$, as the "root" of the tree. Define $\lambda_1 \equiv 1$, and for $j \neq 1$, if $(1 = i_1, i_2, \dots, i_k = j)$ is the unique simple path from 1 to $j$ in $\mathcal{T}$, then let

$$\lambda_j \equiv \prod_{r=1}^{k} \delta_{i_{r-1}, i_r}. \tag{11}$$

A moment's reflection reveals that a different choice of *numeraire*, say $i^* \neq 1$, would produce weights $\lambda_j/\lambda_{i^*}$, $j \in N$. Because the utility representation of $\succsim_S$ that we are seeking is $u_S \equiv (\sum_{i\in S} \lambda_i u_{\bar{i}})/(\sum_{i\in S} \lambda_i)$, the choice of *numeraire* is immaterial.

Given a spanning tree $\mathcal{T}$ of $N$, we say that $S$ is *connected* in $\mathcal{T}$ if $\mathcal{T}_S \equiv \{\overline{ij} \in \mathcal{T}\colon\ i, j \in S\}$ is a spanning tree of $S$. Let $\mathcal{C}$ denote the collection of connected coalitions in $\mathcal{T}$. Singleton coalitions are always connected; a pair $\overline{ij}$ is connected if and only if $\overline{ij} \in \mathcal{T}$. Although the number of connected coalitions will depend on the form of $\mathcal{T}$,[9] the grand coalition $N$ is always connected.

Certain non-degeneracy conditions are required for the aggregation procedure to work. In addition to bilateral agreements, we will need that all triplets have independent utility functions. We say that $\mathcal{T}$ *is non-degenerate* if $d(u_{\bar{i}}, u_{\bar{j}}, u_{\bar{k}}) = 3$ for any $\overline{ijk} \in \mathcal{C}$; and $N$ is *non-degenerate* if $d(u_{\bar{i}}, u_{\bar{j}}, u_{\bar{k}}) = 3$ for any $\overline{ijk} \subseteq N$. Of course, if $N$ is non-degenerate, then so is any spanning tree of $N$. Note that we need $m \geqslant 3$ to have non-degeneracy, and that for any such $m$, non-degeneracy is a "generic" property. This condition is much weaker than the Independent Prospects condition implicit in Harsanyi's aggregation theorem.[10]

**Theorem 4.** *Assume $n-1$ pairs of individuals reach bilateral agreements $\delta_{i,j}$, for $\overline{ij} \in \mathcal{T}$, where $\mathcal{T}$ is a non-degenerate spanning tree of $N$. Use $\delta_{i,j}$ in* (11) *to calculate $(\lambda_1, \dots, \lambda_n)$. If the Extended Pareto Rule holds, then* (*a*) *for all $S \in \mathcal{C}$, $\succsim_S$ is complete and has utility*

$$u_S \equiv \left(\sum_{i\in S} \lambda_i u_i\right) \Big/ \left(\sum_{i\in S} \lambda_i\right). \tag{12}$$

*Moreover,* (*b*) *for all $\overline{ik} \subseteq N$, $u_{\overline{ik}} \in U_{\overline{ik}}$, so that if $\succsim_{\overline{ik}}$ is complete, then it necessarily has utility $u_{\overline{ik}}$.*

---

9 Consider the two extreme examples: a line tree $\mathcal{T}^\ell = \{\{i-1, i\}\colon\ i = 2, \dots, n\}$, and a star tree $\mathcal{T}^* = \{\{1, i\}\colon\ i = 2, \dots, n\}$. In $\mathcal{T}^\ell$ there are $n(n+1)/2$ connected coalitions, which is small with respect to $2^n - 1$, the total number of non-empty coalitions. In $\mathcal{T}^*$, in addition to the singletons, a coalition is connected iff it contains 1, and there are $2^{n-1} - 1$ such non-empty coalitions, yielding $2^{n-1} + n - 1$ connected coalitions. The fraction of connected coalitions tends to $1/2$ as $n$ increases.
The number of spanning trees, by Cayley's formula, is $n^{n-2}$.

10 Independent Prospects demands that for all distinct $i, j, k \in N$, there exist prospects $x, y \in \mathcal{M}$ such that $x \succ_i y$, $y \succ_j x$, and $x \sim_k y$. This amounts to demanding $d(u_{\bar{1}}, u_{\bar{2}}, \dots, u_{\bar{n}}) = n$, which can hold only if $m \geqslant n$. In particular, the example where $\mathcal{M}$ is a list of $m+1$ public projects cannot be handled under Harsanyi unless $m \geqslant n$. Here, we just require Independent Prospects for triples of individuals, which can be satisfied when $m \geqslant 3$. For more on the Independent Prospects condition, see Weymark (1991).

It is clear from Theorem 4 that if all pairs have complete preferences, and $N$ is non-degenerate, then *all* coalitions will also have complete preferences. This establishes claim **A** using minimal premises regarding linear independence. Of course, the weights or utility comparison rates can be calculated as before using any chosen spanning tree. Moreover, the choice of spanning tree is immaterial.[11]

One would expect the $(b)$ part of the theorem to hold for all coalitions, and not just pairs. However, an example can be devised so that under the assumptions of Theorem 4, $u_S \notin U_S$ for some $S \notin \mathcal{C}$. Such examples are pathological, in the sense that $u_S \in U_S$ for all $S \notin \mathcal{C}$ is a generic property.

## 4. Discussion

An interesting observation is that the utility comparison rates can be extended to coalitions. Thus, $\delta_{A,B} \equiv (\sum_{i \in B} \lambda_i)/(\sum_{i \in A} \lambda_i)$ is the utility comparison rate between coalition $A$ and $B$ so that $u_{A \cup B} = (u_A + \delta_{A,B} u_B)/(1 + \delta_{A,B})$. This fact is verified in the proof of Lemma 8.

Having the weights $\lambda_i$ invites us to modify the scales of the corresponding utilities $u_S$ so as to drive all the utility comparison rates to 1. Such an *additive representation* is readily obtainable if we set, for all $S \subseteq N$, $\hat{u}_S \equiv (\sum_{i \in S} \lambda_i) u_S$. It follows that for any two disjoint coalitions $A$ and $B$, $\hat{u}_{A \cup B} = \hat{u}_A + \hat{u}_B$, and so $\hat{u}_S = \sum_{i \in S} \hat{u}_i$. Recall that individual 1 was chosen as a *numeraire* to compute the individual weights $\lambda_i$ in (11). As a consequence, the additive representation expresses all the utilities in the units of individual 1. If we want to use the utility units of some other individual $i^* \neq 1$, it suffices to re-scale each $u_S$ by the factor $\delta_{i^*,1} = 1/\lambda_{i^*}$.

### *4.1. Four individuals and Desargues' theorem*

In the same way that the Extended Pareto Rule construction of Section 2.5 was connected with Ceva's theorem, the case of four individuals is connected to another classical result in geometry. For expository purposes we maintain the affine plane $W$ as in Fig. 2 with $m = 3$ prospect dimensions and $n = 4$ individuals. The fourth individual has utility $u_{\bar{4}} \in W$ such that $d(u_{\bar{2}}, u_{\bar{3}}, u_{\bar{4}}) = 3$ (see Fig. 3). Let $\mathcal{T} = \{\overline{12}, \overline{23}, \overline{34}\}$ be the spanning tree of bilateral agreements.

Using our "no arbitrage" construction of Section 2.5 with $u_{\bar{2}}, u_{\bar{3}}, u_{\bar{4}}, \delta_{2,3}$, and $\delta_{3,4}$ produces a complete preference $\succsim_{\overline{234}}$, with $u_{\overline{234}}$ as the intersection of $u_{\bar{2}} u_{\overline{34}}$ and $u_{\overline{23}} u_{\bar{4}}$. Because we have complete preferences for $\succsim_{\overline{123}}$, we obtain a complete preference $\succsim_{\overline{1234}}$ with $u_{\overline{1234}}$ given by the intersection of $u_{\overline{123}} u_{\bar{4}}$ and $u_{\bar{1}} u_{\overline{234}}$. However, there is a third segment available, namely $u_{\overline{12}} u_{\overline{34}}$. Moreover, the two applications of the "no arbitrage" construction yield $u_{\overline{13}} \in U_{\overline{13}}$ and $u_{\overline{24}} \in U_{\overline{24}}$. Consequently, if $\succsim_{\overline{13}}$ and $\succsim_{\overline{24}}$ were complete, then the segment $u_{\overline{13}} u_{\overline{24}}$ would also be available. But notice that it is impossible to have consistent and complete preferences unless these four segments are *concurrent*, i.e., they have a common point of intersection. This difficulty can be addressed in geometric terms by means of Desargues' theorem (Field and Gray, 1997, pp. 130–131).

[11] To verify that $u_S$ does not depend on the choice of spanning tree suffices to check that if $\hat{\lambda}_i$ and $\lambda_i$ are the weights computed as in (11) using spanning trees $\hat{\mathcal{T}}$ and $\mathcal{T}$, then $\hat{\lambda}_i = \lambda_i$. Clearly $\hat{\lambda}_1 = \lambda_1 = 1$, and for $i \neq 1$, let $(1 = i_1, i_2, \ldots, i_\ell = i)$ be the unique path between 1 and $i$ in $\hat{\mathcal{T}}$. By Theorem 4$b$, all pairs $\overline{i_{r-1} i_r} \in \hat{\mathcal{T}}$ have complete preferences with utilities $u_{\overline{i_{r-1} i_r}} = (\lambda_{i_{r-1}} u_{\overline{i_{r-1}}} + \lambda_{i_r} u_{\overline{i_r}})/(\lambda_{i_{r-1}} + \lambda_{i_r})$. Thus, $\delta_{i_{r-1}, i_r} = \lambda_{i_r}/\lambda_{i_{r-1}}$ and $\hat{\lambda}_i = \prod_{r=1}^{\ell} \delta_{i_{r-1}, i_r} = \prod_{r=1}^{\ell} (\lambda_{i_r}/\lambda_{i_{r-1}}) = \lambda_i$.

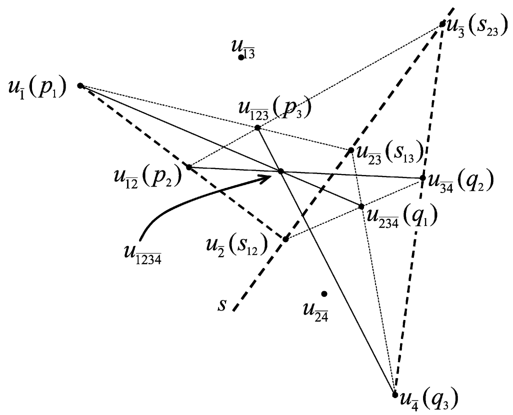


Fig. 3. Desargues' theorem.

**Theorem 5** *(Desargues 1648). Let $p_i$ and $q_i$, for $i = 1, 2, 3$, be two sets of points in a vector space satisfying $p_i \neq q_i$ $(i = 1, 2, 3)$, and that the lines $p_i p_j$ and $q_i q_j$ meet at $s_{ij}$, $1 \leqslant i < j \leqslant 3$. Then, the three lines defined by the points $p_i q_i$, $i = 1, 2, 3$ are concurrent if and only if the three points $s_{ij}$, $1 \leqslant i < j \leqslant 3$, are collinear.*

Figure 3 represents Desargues' theorem as applied to

$$\left.\begin{matrix} p_1 = u_{\bar{1}} & p_2 = u_{\overline{12}} & p_3 = u_{\overline{123}} \\ q_1 = u_{\overline{234}} & q_2 = u_{\overline{34}} & q_3 = u_{\bar{4}} \end{matrix}\right\} \quad \Rightarrow \quad s_{12} = u_{\bar{2}} \quad s_{13} = u_{\overline{23}} \quad s_{23} = u_{\bar{3}}.$$

By *EPR*, $s_{13} \in s_{12}s_{23}$ so that the line segments $u_{\bar{1}}u_{\overline{234}}$, $u_{\overline{12}}u_{\overline{34}}$, and $u_{\overline{123}}u_{\bar{4}}$ are concurrent and $u_{\overline{1234}}$ is well defined. To see that $u_{\overline{1234}} \in u_{\overline{13}}u_{\overline{24}}$, declare $p_2' = u_{\overline{13}}$ and $q_2' = u_{\overline{24}}$, and maintain the other four points. The desired conclusion follows from $s_{13}' = u_{\overline{23}} \in u_{\bar{3}}u_{\bar{2}} = s_{12}'s_{23}'$.[12]

### *4.2. Individuals with trivial preferences*

Individuals with trivial preferences, i.e., $u_i = 0$, find all pairs of prospects indifferent. These individuals can be included in the formulation, provided the following precaution is considered: if $N^*$ is the coalition of individuals with non-trivial preferences, then choose $\mathcal{T}$ in Theorem 4 so that $N^* \in \mathcal{C}$ and $\mathcal{T}_{N^*}$ is non-degenerate.

Both claims are verified by observing that Lemma 7 holds if $u_{\bar{1}} = 0$ and $u_{\bar{2}} \neq 0$, or if $u_{\bar{1}} = u_{\bar{2}} = 0$. However, Lemma 7 fails if $u_{\bar{1}} \neq 0$, $u_{\bar{3}} \neq 0$, and $u_{\bar{2}} = 0$.

### *4.3. Trivial group preferences, stability and Minimal Consensus*

Theorem 4 encompasses the curious case where the group (or some coalition) exhibits total indifference while the individuals have non-trivial preferences.

[12] For the use of Desargues' theorem with larger coalitions, consider three lines given by $u_{A_i}u_{S\backslash A_i}$, $i \in \{1, 2, 3\}$. For $1 \leqslant i < j \leqslant 3$, suppose that $A_i \subset A_j$, and that coalitions $A_i$, $S\backslash A_i$, $A_j\backslash A_i$ have complete preferences. Then, letting $p_i = A_i$ and $q_i = S\backslash A_i$, produces $s_{ij} = A_j\backslash A_i$. That $s_{13} \in s_{12}s_{23}$ follows from $A_3\backslash A_1 = (A_2\backslash A_1) \cup (A_3\backslash A_2)$ and *EPR*.

**Example 6.** Let $N$ be non-degenerate. *EPR* is consistent with trivial preferences for some $S \subseteq N$.

Let $m = 3$, $N = \{1, 2, 3, 4, 5\}$, and individual utilities given by $u_{\bar{1}} = (0, 2, 0)$, $u_{\bar{2}} = (2, 0, 0)$ $u_{\bar{3}} = (-1, -1, 1)$, $u_{\bar{4}} = (-1, -1, -1)$, and $u_{\bar{5}} = (-1, 0, 0)$. Let $\mathcal{T} = \{\overline{12}, \overline{23}, \overline{34}, \overline{45}\}$ and $\delta_{i,j} = 1$, $1 \leqslant i < j \leqslant 5$, so that $u_S = (\sum_{i \in S} u_i)/|S|$. It follows that $u_{\overline{1234}} = 0$ and $u_{\overline{12345}} = u_{\bar{5}}/5 \neq 0$. To determine $\gtrsim_{\overline{1234}}$, observe that any partition $\{A, B, C\}$ of $\overline{1234}$ satisfies (19), but exhibits $d(u_A, u_B, u_C) < 3$. This shows that *EPR* is compatible with a trivial $\gtrsim_{\overline{1234}}$. The proof of Theorem 4 shows that $u_{\overline{1234}} = 0$ is not a problem to establish that $N$ has complete preferences with utility $u_{\overline{12345}}$.

When $u_S = 0$, the preference $\gtrsim_S$ is extremely unstable. Had some individual utility been slightly different, say $u'_{\bar{i}} = u_{\bar{i}} + u$, for some $u \neq 0$, then the corresponding group preference would have had utility $u'_S = u$. However, the choice $u''_{\bar{i}} = u_{\bar{i}} - u$ would produce $u''_S = -u$, i.e., exactly the opposite preference. This unstable behavior is ruled out by imposing the condition of *Minimal Consensus*: there exist two prospects $x, y \in \mathcal{M}$ such that for *all* $i \in N$, $x \succ_{\bar{i}} y$. Clearly, Minimal Consensus and (4) imply $x \succ_S y$, and no coalition has a trivial preference.

### 4.4. Continuity under Minimal Consensus

Non-degeneracy of $N$ is a "generic" property whenever $m \geqslant 3$, i.e., it holds for an open dense set in the space $\mathbb{R}^{mn}$ of *individual profiles* $(u_{\bar{i}})_{i \in N}$. Intuitively, if we choose $n$ utilities at random from $\mathbb{R}^m$, then the probability that any three of them are linearly dependent is zero. This observation suggests extending our results to degenerate individual profiles by using continuity.

Suppose that $(u_{\bar{i}})_{i \in N}$ is a degenerate individual profile, i.e., the non-degeneracy condition of Theorem 4 is not met. Fix $n - 1$ bilateral agreements forming a spanning tree. If $m \geqslant 3$, one can construct a sequence $(u_{\bar{i},k})_{i \in N}$ of non-degenerate individual profiles, and use Theorem 4 to find a sequence of $(u_{S,k})_{S \subseteq N}$, with $u_{S,k}$ given by the $n - 1$ bilateral agreements and (12). By continuity, if $(u_{\bar{i},k})_{i \in N} \to (u_{\bar{i}})_{i \in N}$, then $(u_{S,k})_{S \subseteq N} \to (u_S)_{S \subseteq N}$, where $u_S$ are the utilities calculated using $(u_{\bar{i}})_{i \in N}$, the bilateral agreements, and (12). In summary, this continuity argument extends Theorem 4 to degenerate domains, provided that we are willing to accept the following *Continuity Condition*:

$$\text{If} \quad \gtrsim_{S,k} \text{ is complete and } u_{S,k} \to u_S, \quad \text{then} \quad \gtrsim_S \text{ is complete and has utility } u_S.$$

The Continuity Condition is meaningful whenever $u_S \neq 0$. If $u_S \neq 0$ and $u_{S,k} \to u_S$, then the domination cone $D_k$ associated with $u_{S,k}$ approaches $D_S$, the domination cone associated with $u_S$. This means that if $x \gtrsim_S y$, then there is some $k_0$ such that $x \gtrsim_{S,k} y$ for all $k \geqslant k_0$. However, if $u_S = 0$, then the sequence $u_{S,k} = u/k$, for some $u \neq 0$ satisfies $u_{S,k} \to u_S$, but $\gtrsim_{S,k}$ is the non-trivial preference with utility $u$, whereas $\gtrsim_S$ is trivial. Thus, $\gtrsim_{S,k}$ does not converge to $\gtrsim_S$ in terms of preference. [13] Under Minimal Consensus there are no coalition with trivial preferences, $u_S \neq 0$ for all $S$, and this assures that the continuity condition is meaningful.

[13] It is illustrative to examine the corresponding preference cones: the preference cones of $\gtrsim_{S,k}$ are identical to the half space with normal $u \neq 0$, but this sequence of cones does not converge to $\mathbb{R}^m$, the preference cone of the trivial preference $\gtrsim_S$.

### 4.5. Masters and servants

It is interesting to explore a variation of the strong condition (4) in *EPR*, in which individuals are treated in an asymmetric way. An example will be illustrative. Let $N = \{1, 2, 3, 4\}$ and $\mathcal{T} = \{\overline{12}, \overline{23}, \overline{34}\}$. We still impose the weak condition (3) in *EPR* to all disjoint coalitions, but now reserve the strong condition (4) to certain pairs of individuals as follows: if $1 \leqslant i < j \leqslant 4$, then

$$\text{for all } x, y \in \mathcal{M}, \quad x \succ_{\bar{i}} y, x \succsim_{\bar{j}} y \Rightarrow x \succ_{\overline{ij}} y. \tag{13}$$

Compared to (4), (13) imposes less restrictions on the pair preference $u_{\overline{ij}}$. For example, $i < j$, $x \succ_{\bar{j}} y$, and $x \precsim_{\bar{i}} y$, we now can have $x \sim_{\overline{ij}} y$. This means that $\delta_{i,j} = 0$ and $u_{\overline{ij}} = u_{\bar{i}}$ is possible, i.e., $i$ may prevail over $j$ in the bilateral agreement $\succsim_{\overline{ij}}$. We may think of $i$ as a master and $j$ as $i$'s servant. For example, let $\delta_{2,3} = 0$ and $\delta_{i,j} = 1$ for all other pairs in $\mathcal{T}$ so that 2 dominates 3. Noting that Lemma 7 encompasses $u_{\overline{23}} = u_{\bar{2}}$ whenever $u_{\overline{34}} \neq u_{\bar{4}}$, we find that *EPR* produces that both 1 and 2 *dominate* 3 and 4. Thus, if we require completeness of the pair preference, then we obtain that *EPR* implies the following utilities

$$\begin{aligned} &u_{\overline{13}} = u_{\overline{14}} = u_{\overline{134}} = u_{\bar{1}}, \\ &u_{\overline{23}} = u_{\overline{24}} = u_{\overline{234}} = u_{\bar{2}}, \quad \text{and} \\ &u_{\overline{123}} = u_{\overline{124}} = u_{\overline{1234}} = u_{\overline{12}}. \end{aligned}$$

Taking $i = 1$ as *numeraire*, Formula (11) gives $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = \lambda_4 = 0$, which produces the correct utilities for all coalitions except for $u_{\overline{34}} = (u_{\bar{3}} + u_{\bar{4}})/2 \neq 0$. This calls for the following modification in the procedure to compute a given $u_S$. First, choose a *numeraire* individual in $S$ who is undominated in $S$, and compute $\lambda_{i,S}$ as in (11) for all $i \in S$. Then, (11) yields

$$u_S \equiv \left(\sum_{i \in S} \lambda_{i,S} u_i\right) \Big/ \left(\sum_{i \in S} \lambda_{i,S}\right). \tag{14}$$

The example could be extended as follows. First we establish a relation of dominance between pairs of individuals, assumed irreflexive and acyclic. Then we impose (13) to all pairs $i$ and $j$ such that $i$ dominate $j$. The application of *EPR* after assuming complete preferences for all the pairs produces complete preferences for all coalitions, with utilities computed as in (14). The set of individuals divides itself in hierarchical classes of masters and servants, with $\lambda_{i,S} = 0$ if $S$ contains some individual whose class is higher than $i$'s, and $\lambda_{i,S} > 0$ otherwise.

## 5. Conclusions and extensions

Our theory naturally leads to an interpretation of preference aggregation as a process that begins at the individual level, where complete orderings are assumed. If we recognize the ability of pairs to form complete pair orderings, i.e., establish terms of utility comparison, then *EPR* dictates consistency conditions that build complete orderings from smaller coalitions to larger coalitions. Consequently, we discover that once the problem of welfare comparisons is resolved at a pair level, then it is resolved for the group at large.[14]

We assumed a finite-dimensional prospect space for reasons for simplicity. The extension to infinite-dimensional spaces is quite direct, if we bear in mind that the preliminary section

[14] See Elster and Roemer (1991) for a collection of articles that discuss the problem of interpersonal comparisons of welfare, and also Shapley (1988).

of Shapley and Baucells (1998) articulates the theory of incomplete preferences in such large spaces. One also imagines the extension to countably many individuals once the natural definitions using limits are in place. More challenging seems the extension to uncountable many non-atomic individuals, as Aumann and Shapley (1974) accomplished in the context of cooperative game theory.

Regarding the interpretation of cardinal utility as representing preferences over lotteries, Shapley and Shubik (1982) and Shapley (1975) emphasizes the ability of a cardinal utility scale to represent strength or intensity of preference.[15] In any case, cardinality of the group preference then gives us the possibility of aggregating and averaging individual intensities of preference. Thus, it is convenient to develop the theory of *incomplete* strength of preference. With this in mind, one could introduce a quarternary relation, $(x, y)_{\bar{1}} \gtrsim_{\overline{12}} (z, w)_{\bar{2}}$, as the basis to express interpersonal comparisons of strength of preference. Such a relation could be incomplete, and hopefully represented by a cone of utility functions. Thus, a cardinal framework for group utility could be obtained without involving lotteries.

We have provided a result under minimal conditions. If one is willing to assume that individual utilities are linearly independent, then the proof of Theorem 4 can be shortened, and embedded in a more general result (Baucells and Shapley, 2007).

Two further papers are planned. In the first, we use the current framework, that encompasses both complete and incomplete group preferences, and consider pair agreements that exhibit some degree of incompleteness. The Extended Pareto Rule then determines an incomplete group preference, but not as incomplete as the preference derived from the usual Pareto rule. We attempt to measure the degree of incompleteness of such partial orderings by introducing an invariant measure on the interior of any given simplicial "conic section" of the cone of utility functions—invariant, that is, under the group of all projective transformations that hold fixed the vertices of the given simplex. Then, we explore how the degree of incompleteness in the pair-preferences restricts the incompleteness of the group preference. This enables us to study the convergence of incomplete preferences to complete preferences as $n$ grows, exploiting the very rapid growth of the number of partitions of $N$.

In the second paper we will further develop the more general non-cardinal approach described in claim **A** of the Introduction. As illustrated in a three-individual example in Shapley and Shubik (1982, p. 66), a complete group preference is derived from two pair preferences and the Extended Pareto Rule, both the individual and social preferences being expressed by means of ordinal utilities. Thus, our cardinal setting appears as a special framework to illustrate a more general phenomenon. While the cardinal setting allows us to use a convenient representation of incomplete preferences in terms of convex cones, the ordinal extension seems to require a totally different conceptual machinery. This, we hope, will shed light on the aggregation procedure leading to non-linear welfare functions.

## Acknowledgment

This paper is part of Manel's PhD dissertation. This author is grateful to IESE Business School for the financial support received during that time. For helpful comments, we are grate-

[15] Under natural assumptions, the cardinal utility function representing intensities of preference coincides with the cardinal utility function representing choices over lotteries. See Sarin (1982) for a treatment of this point in the context of Subjective Expected Utility, and Dyer and Sarin (1979) for non-linear representations of group strength of preference, that can be distinguished from non-linear group utilities for choices over lotteries.

ful to the participants of the IX International Game Theory Conference at Stony Brook; the UCLA workshop in Economic Theory (March 1999); the Southern California Operations Research/Management Science Annual Meeting (June 1999); the Economic Theory Seminar at the Universitat Autònoma de Barcelona (October 1999); the Decision Analysis seminar at the Fuqua School of Business (Duke University); the Public Choice Society meeting (Charleston, March 2000); and the 8th world congress of the Econometric Society. The first part of this paper was presented in the 13th International Symposium on Mathematical Programming.

## Appendix A

**Proof of Theorem 2.** By the definition of preference cone, (3)is equivalent to $D_A \cap D_B \subseteq D_{A\cup B}$; and by the properties of polar cones (Rockafellar, 1970, pp. 125 and 150), $D_A \cap D_B \subseteq D_{A\cup B} \Leftrightarrow D^*_{A\cup B} \subseteq (D_A \cap D_B)^* = \mathrm{Co}(D^*_A, D^*_B) = (D^*_A \cup D^*_B)^{**}$. Because $(\cdot)^{**}$ is the closure of the set of positive multiples of convex combinations of a given set, and $(D^*_A \cup D^*_B)$ is already a closed cone, we have that $(D^*_A \cup D^*_B)^{**} = \mathrm{Co}(D^*_A \cup D^*_B)$, whence (3) is equivalent to $D^*_{A\cup B} \subseteq \mathrm{Co}(D^*_A \cup D^*_B)$. A moment's reflection reveals that this last inclusion, together with the condition

$$\text{If } U_{A\cup B} = \{0\}, \quad \text{then} \quad 0 \in \mathrm{Co}_{A,B}, \tag{15}$$

is equivalent to (7). But (4) implies (15). Thus, if $0 \notin \mathrm{Co}_{A,B}$, then $\mathrm{Co}_{A,B}$ is a pointed cone, i.e., both $\succsim_A$ and $\succsim_B$ contain strict preferences, so that $\succsim_{A\cup B}$ also contains strict preferences and $U_{A\cup B} \neq \{0\}$. Consequently, [(3), (4)⇒(7)], and [(7)⇒(3)].

[(3), (4)⇒(8)] Suppose that (8) fails so that $U_{A\cup B} \subseteq \mathrm{Co}_{A,B} \backslash \mathrm{Co}^{ri}_{A,B}$. We enlarge $\mathrm{Co}_{A,B}$ and define the full dimensional cone $K_{A,B} = \mathrm{Co}_{A,B} \times Sp(\mathrm{Co}_{A,B})^{\perp}$, which is the Cartesian product of $\mathrm{Co}_{A,B}$ with the subspace orthonormal to $Sp(\mathrm{Co}_{A,B})$. Of course, $\mathrm{Co}^{ri}_{A,B}$ is contained in the interior of $K_{A,B}$. Because $U_{A\cup B}$ is convex and contained in the boundary of $\mathrm{Co}_{A,B}$, it has dimension strictly less than that of $\mathrm{Co}_{A,B}$ and $K_{A,B}$. It follows that there is a hyperplane $H$ containing $U_{A\cup B}$ and supporting $K_{A,B}$. Full dimensionality ensures that $H$ does not intersect the interior of $K_{A,B}$. Thus, $H$ supports $\mathrm{Co}_{A,B}$ and does not intersect $\mathrm{Co}^{ri}_{A,B}$. If $x - y$ is some normal vector of $H$, then $u(x-y) = 0$ for all $u \in U_{A\cup B}$, and $u(x-y) \geqslant 0$ for all $u \in \mathrm{Co}_{A,B}$. Because $\mathrm{Co}_{A,B}$ is non-empty, $\mathrm{Co}^{ri}_{A,B}$ is non-empty: there is a $u \in \mathrm{Co}_{A,B}$ such that $u(x-y) > 0$, and we can find one such $u$ in either $U_A$ or $U_B$, say $U_A$. Thus, $x \succ_A y$ and $x \succsim_B y$, but $x \sim_{A\cup B} y$, a contradiction of (4).

[(7), (8)⇒(4)] If for some $x, y \in \mathcal{M}$, $x \succ_A y$ and $x \succsim_B y$, then $u(x-y) \geqslant 0$ for all $u \in \mathrm{Co}_{A,B}$, and $u_A(x-y) > 0$ for some $u_A \in U_A$. By (8), let $u^* \in U_{A\cup B} \cap \mathrm{Co}^{ri}_{A,B}$ so that for some $u' \in \mathrm{Co}_{A,B}$ and $\alpha \in (0,1)$, $u^* = (1-\alpha)u_A + \alpha u'$. By (7) and the argument above $u(x-y) \geqslant 0$ for all $u \in U_{A\cup B}$, and $u^*(x-y) > 0$, showing that $x \succ_{A\cup B} y$. □

In order to prove Theorem 4 we need two lemmas. The first is basically Ceva's theorem under minimal assumptions of degeneracy. Indeed, degeneracies may preclude the aggregation procedure discussed for the $n = 3$ individual case. For example, if $u_{\bar{1}}$, $u_{\bar{2}}$, and $u_{\bar{3}}$ are collinear, then the segments $u_{\bar{1}}u_{\overline{23}}$ and $u_{\overline{12}}u_{\bar{3}}$ are parallel and the procedure does not work. The following lemma requires a condition weaker than $d(u_{\bar{1}}, u_{\bar{2}}, u_{\bar{3}}) = 3$.

**Lemma 7.** *Assume that EPR holds, and consider bilateral agreements $\succsim_{\overline{12}}$ and $\succsim_{\overline{23}}$ with utility comparison rates $\delta_{1,2}$ and $\delta_{2,3}$ that determine $u_{\overline{12}}$ and $u_{\overline{23}}$. Let $\delta_{1,3} \equiv \delta_{1,2}\delta_{2,3}$.*

(a) *If either $u_{\bar{1}} \notin Sp(u_{\bar{3}}, u_{\overline{12}})$ or $u_{\bar{3}} \notin Sp(u_{\bar{1}}, u_{\overline{23}})$, then $\succsim_{\overline{123}}$ is complete and has utility $u_{\overline{123}} = (u_{\bar{1}} + \delta_{1,2}u_{\bar{2}} + \delta_{1,3}u_{\bar{3}})/(1 + \delta_{1,2} + \delta_{1,3})$.*

(b) *If, moreover, $u_{\bar{2}} \notin Sp(u_{\bar{1}}, u_{\bar{3}})$, then $u_{\overline{13}} \in U_{\overline{13}}$, so that if $\succsim_{\overline{13}}$ is complete, then it necessarily has utility $u_{\overline{13}} = (u_{\bar{1}} + \delta_{1,3}u_{\bar{3}})/(1+\delta_{1,3})$.*

**Proof of Proposition 7.** (a) By (7), for any $u^* \in U_{\overline{123}}$ there are $\alpha, \beta \in [0,1]$ and $\lambda_\alpha, \lambda_\beta > 0$ such that

$$\lambda_\alpha\big[(1-\alpha)u_{\bar{1}} + \alpha u_{\overline{23}}\big] = u^* = \lambda_\beta\big[(1-\beta)u_{\overline{12}} + \beta u_{\bar{3}}\big]. \tag{16}$$

From the definitions of $u_{\overline{12}}$ and $u_{\overline{23}}$, $u_{\overline{12}} = \{u_{\bar{1}} + \delta_{1,2}[u_{\overline{23}} + \delta_{2,3}(u_{\overline{23}} - u_{\bar{3}})]\}/(1+\delta_{1,2})$. Substituting this expression in the right-hand side of (16) produces an expression involving only $u_{\bar{1}}$, $u_{\overline{23}}$, $u_{\bar{3}}$. Because $u_{\bar{3}} \notin Sp(u_{\bar{1}}, u_{\overline{23}})$ (in particular $u_3 \neq 0$), we equate the coefficients of $u_{\bar{3}}$ in the modified expression (16) to conclude that $\beta = \delta_{1,3}/(1+\delta_{1,2}+\delta_{1,3})$. Replacing $\beta$ and $u_{\overline{12}} = (u_{\bar{1}} + \delta_{1,2}u_{\bar{2}})/(1+\delta_{1,2})$ in the right-hand side of (16) yields $u^* = \lambda_\beta u_{\overline{123}}$. We can construct a similar argument if $u_{\bar{1}} \notin Sp(u_{\bar{3}}, u_{\overline{12}})$ by replacing $u_{\overline{23}}$ for an expression that involves $u_{\bar{3}}$, $u_{\overline{12}}$, and $u_{\bar{1}}$. Thus, $\succsim_{\overline{123}}$ has utility $u_{\overline{123}}$ and $(a)$ follows.

(b) Let $\succsim_{\overline{13}}$ be complete with utility $u^{**}$. By Corollary 3, there is some $\alpha' \in (0,1)$ and $\lambda'_\alpha > 0$ such that $u_{\overline{123}} = \lambda'_\alpha[(1-\alpha')u_{\bar{2}} + \alpha' u^{**}]$; similarly, some $\beta' \in (0,1)$ and $\lambda'_\beta > 0$ such that $u^{**} = \lambda'_\beta[(1-\beta')u_{\bar{1}} + \beta' u_{\bar{3}}]$. Thus,

$$\lambda'_\alpha\big[(1-\alpha')u_{\bar{2}} + \alpha'\lambda'_\beta\big[(1-\beta')u_{\bar{1}} + \beta' u_{\bar{3}}\big]\big] = u_{\overline{123}} = \frac{u_{\bar{1}} + \delta_{1,2}u_{\bar{2}} + \delta_{1,3}u_{\bar{3}}}{1+\delta_{1,2}+\delta_{1,3}}. \tag{17}$$

That $u_{\bar{2}} \notin Sp(u_{\bar{1}}, u_{\bar{3}})$ implies $\lambda'_\alpha(1-\alpha') = \delta_{1,2}/(1+\delta_{1,2}+\delta_{1,3})$ and $\lambda'_\alpha\alpha' u^{**} = (u_{\bar{1}} + \delta_{1,3}u_{\bar{3}})/(1+\delta_{1,2}+\delta_{1,3}) = u_{\overline{13}}$. Because $\alpha', \lambda'_\alpha > 0$, if $\succsim_{\overline{13}}$ is complete, then it has utility $u_{\overline{13}}$. Upon reflection, this is equivalent to $u_{\overline{13}} \in U_{\overline{13}}$. □

Next, we generalize Lemma 7 as follows:

**Lemma 8.** *For some collection of weights $(\lambda_1, \ldots, \lambda_n) > 0$, consider the utility functions $u_T = (\sum_{i\in T}\lambda_i u_{\bar{i}})/(\sum_{i\in T}\lambda_i)$, for $T \subseteq N$. Let $A, B, C$ be disjoint coalitions and $S = A\cup B\cup C$. The following are consequences of the EPR.*

(a) *Suppose $u_A \notin Sp(u_C, u_{A\cup B})$ [or $u_C \notin Sp(u_A, u_{B\cup C})$]. For $T \in \{A, C, A\cup C, B\cup C\}$, if $\succsim_T$ is complete and has utility $u_T$, then $\succsim_S$ is complete and has utility $u_S$.*

(b) *Suppose $u_B \notin Sp(u_A, u_C)$. For $T \in \{A, B, C, S\}$, if $\succsim_T$ is complete and has utility $u_T$, then $u_{A\cup C} \in U_{A\cup C}$, i.e., if $\succsim_{A\cup C}$ is complete, then it has utility $u_{A\cup C}$.*

**Proof.** To see that the utility comparison rates between two disjoint coalitions $A$ and $B$ is $\delta_{A,B} \equiv (\sum_{i\in B}\lambda_i)/(\sum_{i\in A}\lambda_i)$, consider

$$u_{A\cup B} = \frac{\sum_{i\in A}\lambda_i u_{\bar{i}} + \sum_{i\in B}\lambda_i u_{\bar{i}}}{\sum_{i\in A\cup B}\lambda_i} = \frac{(\sum_{i\in A}\lambda_i)u_A + (\sum_{i\in B}\lambda_i)u_B}{\sum_{i\in A\cup B}\lambda_i} = \frac{u_A + \delta_{A,B}u_B}{1+\delta_{A,B}}. \tag{18}$$

The result then follows from Proposition 7 by using $u_A$, $u_B$, $u_C$, $\delta_{A,B}$, and $\delta_{B,C}$ in place of $u_{\bar{1}}$, $u_{\bar{2}}$, $u_{\bar{3}}$, $\delta_{1,2}$, and $\delta_{2,3}$; and checking that $(u_A + \delta_{A,B}u_B + \delta_{A,C}u_C)/(1+\delta_{A,B}+\delta_{A,C}) = (\sum_{i\in S}\lambda_i u_{\bar{i}})/(\sum_{i\in S}\lambda_i) = u_S$. □

**Proof of Theorem 4.** (a) If $\mathcal{C}$ is the collection of connected coalitions in $\mathcal{T}$, let $\mathcal{C}_r$ indicate the connected coalitions of size $r$. We claim that for $r \geqslant 3$ and $S \in \mathcal{C}_r$, then there is a partition $\{A, B, C\}$ of $S$ such that $\{A, C, A \cup B, B \cup C\} \subseteq \mathcal{C}$ and[16]

$$u_A \notin Sp(u_C, u_{A\cup B}) \quad \text{and} \quad u_C \notin Sp(u_A, u_{B\cup C}). \tag{20}$$

The result easily follows from the claim. If $S \in \mathcal{C}_3$, then the partition of $S$ given by the claim has its elements in $\mathcal{C}_1 \cup \mathcal{C}_2$. Because $\gtrsim_T$ is complete and has utility $u_T$ for all $T \in \mathcal{C}_1 \cup \mathcal{C}_2$, Lemma 8(a) establishes this property for $\gtrsim_S$. Similarly, once this is established for all $T \in \mathcal{C}_\ell$, $\ell < r$, then it also holds for all $S \in \mathcal{C}_r$; the partition $\{A, B, C\}$ of $S$ given by the claim has its members in $\mathcal{C}_\ell$, $\ell < r$, and (20) allow us to apply Lemma 8(a).

We establish the claim by induction. For $r = 3$, let $\overline{ijk} \in \mathcal{C}_3$ and define the partition $\{A, B, C\} = \{\bar{i}, \bar{j}, \bar{k}\}$ of $\overline{ijk}$, so that $\{\bar{i}, \bar{k}, \overline{ij}, \overline{jk}\} \subseteq \mathcal{C}$. $\mathcal{T}$ non-degenerate guarantees (20).

For $r \geqslant 4$, assume that the claim is true for all the coalitions in $\mathcal{C}_\ell$, $\ell < r$. If degeneracies were not a problem, the proof would be as follows. If $S \in \mathcal{C}_r$ and $i \in S$ is a terminal node of $S$,[17] then $S\backslash\bar{i} \in \mathcal{C}_{r-1}$. Let $\{\tilde{A}, \tilde{B}, \tilde{C}\}$ be the partition of $S\backslash\bar{i}$ given by induction, and $j$ the unique adjacent of $i$ in $S\backslash\bar{i}$. The partition $\{A, B, C\}$ of $S$ is defined as follows: if $j \in \tilde{A}$, then use $\{\tilde{A} \cup \bar{i}, \tilde{B}, \tilde{C}\}$; if $j \in \tilde{B}$, then use $\{\tilde{A}, \tilde{B} \cup \bar{i}, \tilde{C}\}$; and if $j \in \tilde{C}$, then use $\{\tilde{A}, \tilde{B}, \tilde{C} \cup \bar{i}\}$. One observes that $\{A, C, A \cup B, B \cup C\} \subseteq \mathcal{C}$ in all three cases. However, condition (20) may fail if $d(u_A, u_B, u_C) < 3$. The remedy consist of first replacing the terminal node $i$ by a connected coalition $R \in \mathcal{C}_1 \cup \mathcal{C}_2$ such that $S\backslash R \in \mathcal{C}$ and $u_{S\backslash R} \neq 0$. If $\{\tilde{A}, \tilde{B}, \tilde{C}\}$ is the partition of $S\backslash R$ given by induction, we ensure condition (20) by choosing which two coalitions to "glue" from $\{\tilde{A}, \tilde{B}, \tilde{C}, R\}$ to produce the partition $\{A, B, C\}$ of $S$.

To find $R$, let $j$ be the node with a maximal number $t(j)$ of terminal adjacent nodes in $\mathcal{C}_r$. If $t(j) = 1$, then let $i$ be this terminal node and define $R = \bar{i}$ if $u_{S\backslash\bar{i}} \neq 0$, and $R = \overline{ij}$ otherwise (because $\lambda_j u_{\bar{j}} \neq 0$, $u_{S\backslash\overline{ij}} \neq 0$). If $t(j) \geqslant 2$, let $i$ and $k$ be two terminal adjacent nodes of $j$ and define $R = \bar{i}$ if $u_{S\backslash\bar{i}} \neq 0$, and $R = \bar{k}$ otherwise (if $u_{S\backslash\bar{i}} = 0$, then $d(u_{\bar{i}}, u_{\bar{j}}, u_{\bar{k}}) = 3$ implies $u_{S\backslash\bar{k}} \neq 0$). Thus, $R \in \mathcal{C}$, $S\backslash R \in \mathcal{C}_\ell$ for some $\ell \geqslant 3$ (note that when $r = 4$, a non-degenerate $\mathcal{T}$ guarantees that $R = \bar{i}$ and $S\backslash R \in \mathcal{C}_3$), and $u_{S\backslash R} \neq 0$. By induction, let $\{\tilde{A}, \tilde{B}, \tilde{C}\}$ be the partition of $S\backslash R$ satisfying the claim. Because of the symmetric role of $\tilde{A}$ and $\tilde{C}$, we can assume without loss of generality that either $R \cup \tilde{A} \in \mathcal{C}$ or $R \cup \tilde{B} \in \mathcal{C}$ and define the partition $\{A, B, C\}$ of $S$ as follows:

| $R \cup \tilde{A} \in \mathcal{C}$ | $A$ | $B$ | $C$ | Case |
|---|---|---|---|---|
| (1) | $R$ | $\tilde{A}$ | $\tilde{B} \cup \tilde{C}$ | if $u_{\tilde{C}} \in Sp(u_{\tilde{A}\cup R}, u_{\tilde{B}\cup\tilde{C}})$ |
| (2) | $R$ | $\tilde{A} \cup \tilde{B}$ | $\tilde{C}$ | if $u_{\tilde{A}\cup R} \in Sp(u_{\tilde{C}}, u_{\tilde{A}\cup\tilde{B}\cup R})$ |
| (3) | $\tilde{C}$ | $\tilde{B}$ | $\tilde{A} \cup R$ | otherwise. |

16 If Minimal Consensus holds, then we can replace (20) by $d(u_A, u_B, u_C) = 3$. To see this, notice that condition (20) holds if either $d(u_A, u_B, u_C) = 3$ or

$$u_A = -\gamma u_{B\cup C} \neq 0 \quad \text{and} \quad u_C = -\gamma' u_{A\cup B} \neq 0, \quad \text{for some } \gamma, \gamma' > 0. \tag{19}$$

But Minimal Consensus rules out (19).

17 An individual $i \in S$ is *terminal* in a connected coalition $S$ if there is only one $j \in S$ such that $\overline{ij} \in \mathcal{T}_S$.

| $R \cup \tilde{B} \in \mathcal{C}$ | $A$ | $B$ | $C$ | Case |
|---|---|---|---|---|
| (4) | $R$ | $\tilde{B} \cup \tilde{C}$ | $\tilde{A}$ | if $u_{\tilde{C}} \in Sp(u_{\tilde{A}}, u_{\tilde{B}\cup\tilde{C}\cup R})$ |
| (5) | $R$ | $\tilde{A} \cup \tilde{B}$ | $\tilde{C}$ | if $u_{\tilde{A}} \in Sp(u_{\tilde{C}}, u_{\tilde{A}\cup\tilde{B}\cup R})$ |
| (6) | $\tilde{A}$ | $\tilde{B} \cup R$ | $\tilde{C}$ | otherwise. |

Upon examination one confirms that $\{A, C, A \cup B, B \cup C\} \subseteq \mathcal{C}$ holds in all six cases. By construction, (20) holds in cases (3) and (6). We now give the details showing that (20) holds in (1), i.e., that $u_R \notin Sp(u_{\tilde{B}\cup\tilde{C}}, u_{\tilde{A}\cup R})$ and $u_{\tilde{B}\cup\tilde{C}} \notin Sp(u_R, u_{S\setminus R})$. An adaptation of the same argument establishes cases (2), (4) and (5). Recall that by the inductive hypotheses given by (20), both $u_{\tilde{A}} \notin Sp(u_{\tilde{C}}, u_{\tilde{A}\cup\tilde{B}})$ and $u_{\tilde{C}} \notin Sp(u_{\tilde{A}}, u_{\tilde{B}\cup\tilde{C}})$.

If (1) applies, then $u_{\tilde{C}} \in Sp(u_{\tilde{A}\cup R}, u_{\tilde{B}\cup\tilde{C}})$ (see Fig. 4), and so $u_{\tilde{C}} = \alpha u_{\tilde{A}\cup R} + \beta u_{\tilde{B}\cup\tilde{C}}$ for some $\alpha$ and $\beta$. That $u_{\tilde{C}} \notin Sp(u_{\tilde{A}}, u_{\tilde{B}\cup\tilde{C}})$ rules out $\alpha = 0$, and using $u_{\tilde{A}\cup R} = (u_R + \delta_{R,\tilde{A}} u_{\tilde{A}})/(1 + \delta_{R,\tilde{A}})$ as in (18) we write

$$u_R = (1 + \delta_{R,\tilde{A}})(u_{\tilde{C}} - \beta u_{\tilde{B}\cup\tilde{C}} t)/\alpha - \delta_{R,\tilde{A}} u_{\tilde{A}}. \tag{21}$$

Also, $u_{\tilde{A}} \notin Sp(u_{\tilde{C}}, u_{\tilde{A}\cup\tilde{B}})$ is incompatible with $u_{\tilde{B}\cup\tilde{C}} = -\gamma u_{\tilde{A}}$, for some $\gamma \geqslant 0$. Otherwise, we write $(1 + \delta_{\tilde{A},\tilde{B}\cup\tilde{C}}) u_{S\setminus R} = u_{\tilde{A}} + \delta_{\tilde{A},\tilde{B}\cup\tilde{C}} u_{\tilde{B}\cup\tilde{C}} = u_{\tilde{A}}(1 - \gamma\delta_{\tilde{A},\tilde{B}\cup\tilde{C}})$. $u_{S\setminus R} \neq 0$ implies $\gamma\delta_{\tilde{A},\tilde{B}\cup\tilde{C}} \neq 1$ and $u_{\tilde{A}} \in Sp(u_{S\setminus R})$. This, coupled with $U_{S\setminus R} \subseteq \mathrm{Co}(u_{\tilde{C}}, u_{\tilde{A}\cup\tilde{B}})$ contradicts $u_{\tilde{A}} \notin Sp(u_{\tilde{C}}, u_{\tilde{A}\cup\tilde{B}})$.

Now, assume that $u_R \in Sp(u_{\tilde{B}\cup\tilde{C}}, u_{\tilde{A}\cup R})$ so that $u_R = \alpha_1 u_{\tilde{B}\cup\tilde{C}} + \beta_1(u_R + \delta_{R,\tilde{A}} u_{\tilde{A}})$ for some $\alpha_1, \beta_1$. If $\beta_1 \neq 1$, then use (21) to eliminate $u_R$ and obtain $u_{\tilde{C}} \in Sp(u_{\tilde{A}}, u_{\tilde{B}\cup\tilde{C}})$, a contradiction. If $\beta_1 = 1$, then $u_{\tilde{B}\cup\tilde{C}} = -\delta_{R,\tilde{A}} u_{\tilde{A}}/\alpha_1$, contradicting $u_{\tilde{A}} \notin Sp(u_{\tilde{C}}, u_{\tilde{A}\cup\tilde{B}})$.

Similarly, assume that $u_{\tilde{B}\cup\tilde{C}} \in Sp(u_R, u_{S\setminus R})$, so that $u_{\tilde{B}\cup\tilde{C}} = \alpha_2 u_R + \beta_2(u_{\tilde{A}} + \delta_{\tilde{A},\tilde{B}\cup\tilde{C}} u_{\tilde{B}\cup\tilde{C}})$ for some $\alpha_2, \beta_2$. If $\alpha_2 \neq 0$, then use (21) to eliminate $u_R$ and obtain $u_{\tilde{C}} \in Sp(u_{\tilde{A}}, u_{\tilde{B}\cup\tilde{C}})$, a contradiction. If $\alpha_2 = 0$, then $u_{\tilde{B}\cup\tilde{C}} = \beta_2 u_{\tilde{A}}/(1 - \beta_2\delta_{\tilde{A},\tilde{B}\cup\tilde{C}})$, contradicting $u_{\tilde{A}} \notin Sp(u_{\tilde{C}}, u_{\tilde{A}\cup\tilde{B}})$, provided $\beta_2\delta_{\tilde{A},\tilde{B}\cup\tilde{C}} \neq 1$. This same contradiction arises if $\beta_2\delta_{\tilde{A},\tilde{B}\cup\tilde{C}} = 1$ and $u_{\tilde{A}} = 0$.

(b) The result is trivial if $\overline{ik} \in \mathcal{T}$. For $\overline{ik} \notin \mathcal{T}$, let $(i = i_1, \ldots, i_\ell = k)$ be the path in $\mathcal{T}$ between $i$ and $k$, and $T = \cup_{r=2}^{\ell-1} \bar{i}_r \neq \emptyset$. We claim that there is a coalition $R \subseteq T$ such that $R \in \mathcal{C}$, $u_R \notin Sp(u_{\bar{i}}, u_{\bar{k}})$, and $u_{R\cup\overline{ik}} \in U_{R\cup\overline{ik}}$. Indeed, because $\gtrsim_R$ is complete, the result follows from letting $A = \bar{i}$, $B = R$, and $C = \bar{k}$ in Lemma 8(b). To establish the claim we define $R$ as follows. If $u_T \notin Sp(u_{\bar{i}}, u_{\bar{k}})$, then let $R = T$ (this is always the case if $\ell = 3$). On the contrary, if $u_T \in Sp(u_{\bar{i}}, u_{\bar{k}})$, then $\mathcal{T}$ non-degenerate implies $d(u_{\bar{i}}, u_{\bar{i_2}}, u_{\bar{i_3}}) = 3$, so that either $u_{\bar{i_2}} \notin Sp(u_{\bar{i}}, u_{\bar{k}})$ or $u_{\bar{i_3}} \notin Sp(u_{\bar{i}}, u_{\bar{k}})$, the former being always true by $\mathcal{T}$ non-degenerate if $\ell = 4$. Accordingly, let $R = \overline{i_2}$ or $R = \overline{i_2 i_3}$ so that $T \setminus R \neq \emptyset$, $u_T \in Sp(u_{\bar{i}}, u_{\bar{k}})$, and $u_R \notin Sp(u_{\bar{i}}, u_{\bar{k}})$. Having $(1 + \delta_{R,T\setminus R}) u_T = (u_R + \delta_{R,T\setminus R} u_{T\setminus R})$ for some $\delta_{R,T\setminus R} > 0$ implies $u_{T\setminus R} \notin Sp(u_{\bar{i}}, u_{\bar{k}})$. Because $\{R \cup \bar{i}, T\setminus R, \bar{k}, T \cup \overline{ik}\} \subseteq \mathcal{C}$ we use $A = R \cup \bar{i}$, $B = T \setminus R$, and $C = \bar{k}$ in Lemma 8(b) to conclude $u_{R\cup\overline{ik}} \in U_{R\cup\overline{ik}}$.

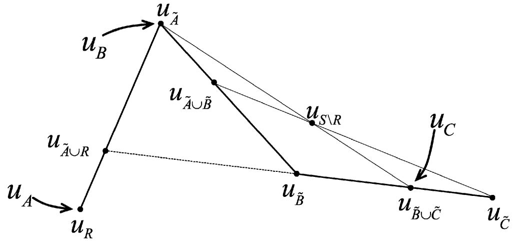


Fig. 4. Identify $A = R$, $B = \tilde{A}$, and $C = \tilde{B} \cup \tilde{C}$ when $u_{\tilde{C}} \in Sp(u_{\tilde{A}\cup R}, u_{\tilde{B}\cup\tilde{C}})$.

Geometrically, $u_{R\cup\bar{i}\bar{k}}$ is found as the intersection of the line segment $u_{R\cup\bar{i}}u_{\bar{k}}$ and the half line $u_{T\cup\bar{i}\bar{k}}u_{T\setminus R}$. Similarly, $u_{\bar{i}\bar{k}}$ is found as the intersection of the line segment $u_{\bar{i}}u_{\bar{k}}$ and the half line $u_{R\cup\bar{i}\bar{k}}u_R$. Because $u_{\bar{i}\bar{k}} = (\lambda_i u_{\bar{i}} + \lambda_k u_{\bar{k}})/(\lambda_i + \lambda_k)$, the utility comparison rate between $i$ and $k$ is given by $\delta_{i,k} = \lambda_k/\lambda_i = \prod_{r=1}^{\ell} \delta_{i_{r-1},i_r}$. □

## References

Arrow, K.J., 1963. Social Choice and Individual Values. Cowles Found. Monogr., vol. 12. Yale Univ. Press.
Aumann, R.J., 1962. Utility theory without the completeness axiom. Econometrica 30, 445–462; and: A correction, Econometrica 32 (1964) 210–212.
Aumann, R.J., Shapley, L.S., 1974. Values of Non-Atomic Games. Princeton Univ. Press, Princeton.
Baucells, M., Sarin, R., 2003. Group decisions with multiple criteria. Manage. Sci. 49 (8), 1105–1118.
Baucells, M., Shapley, L.S., 2007. Multiperson utility: The linearly independent case. IESE Business School. Available at http://webprofesores.iese.edu/mbaucells/.
Bewley, T., 1986. Knightian decision theory: Part i. Discussion paper No. 807. Cowles Foundation, Yale University. Published in: Dec. Econ. Finance 25 (2002).
Brannan, M., Esplen, F., Gray, J.J., 1999. Geometry. Cambridge Univ. Press, Princeton.
Dhillon, A., 1998. Extended Pareto Rules and relative utilitarianism. Soc. Choice Welfare 15, 521–542.
Dhillon, A., Mertens, J.F., 1999. Relative utilitarianism. Econometrica 67 (3), 471–498.
Dubra, J., Maccheroni, F., Ok, E.A., 2004. Expected utility theory without the completeness axiom. J. Econ. Theory 115 (1), 118–133.
Dyer, J.S., Sarin, R.K., 1979. Group preference aggregation rules based on strenght of preference. Manage. Sci. 25 (9), 822–832.
Elster, J., Roemer, J.E. (Eds.), 1991. Interpersonal Comparisons of Well-Being. Cambridge Univ. Press.
Field, J.V., Gray, J.J. (Eds.), 1997. The Geometrical Work of Girard Desargues. Springer-Verlag, New York.
Fishburn, P.C., 1973. A mixture-set axiomatization of conditional subjective utility. Econometrica 41 (1), 1–25.
Harsanyi, J.C., 1955. Cardinal welfare, individualistic ethics, and interpersonal comparisons of utility. J. Polit. Economy 63, 309–321.
Karni, E., 2003. Impartiality and interpersonal comparisons of variations in well-being. Soc. Choice Welfare 21 (1), 95–111.
Luce, R.D., Krantz, D.H., 1971. Conditional expected utility. Econometrica 39 (2), 253–271.
Nau, R., 2006. The shape of incomplete preferences. Ann. Statist. 34 (5), 2430–2448.
Rigotti, L., Shannon, C., 2004. Uncertainty and risk in financial markets. Econometrica 73 (1), 203–243.
Rockafellar, T.R., 1970. Convex Analysis. Princeton Univ. Press, Princeton, NJ.
Sarin, R.K., 1982. Strength of preference and risky choice. Operations Res. 30 (5).
Schmeidler, D., 1969. Competitive equilibria in markets with a continuum of traders and incomplete preferences. Econometrica 37 (4), 578–585.
Sen, A., 1970. Interpersonal aggregation and partial comparability. Econometrica 38, 393–409; and: A correction, Econometrica 40 (1972) 959.
Shapley, L.S., 1975. Cardinal utility from intensity comparisons. RAND report R-1683-NSF, Santa Monica. See also Shubik (1982, Appendix A. 3).
Shapley, L.S., 1988. Utility comparisons and the theory of games. In: Roth, A.E. (Ed.), The Shapley Value. Cambridge Univ. Press, Cambridge, MA, pp. 307–319. Originally appeared in La Decision, Editions du CNRS, Paris, 1969, pp. 251–263.
Shapley, L.S., Baucells, M., 1998. Multiperson utility. Working paper No. 779. Department of Economics, UCLA. Available at http://webprofesores.iese.edu/mbaucells/.
Shapley, L.S., Shubik, M., 1982. Preferences and utility. In: Shubik, M. (Ed.), Game Theory in the Social Sciences: Concepts and Solutions. MIT Press, Cambridge, MA. Originally appeared in Preferences and Utility, RAND report R-904/4-NSF, Santa Monica, 1974, Chapters 4 and 5.
Shubik, M., 1982. Game Theory in the Social Sciences: Concepts and Solutions. MIT Press, Cambridge, MA.
Skiadas, C., 1997. Conditioning and aggregation of preferences. Econometrica 65 (2), 347–367.
Sobel, J., 2001. Manipulation of preferences and relative utilitarianism. Games Econ. Behav. 37, 196–215.
Vind, K., 2000. Von Neumann–Morgenstern preferences. J. Math. Econ. 33, 109–122.
Von-Neumann, J., Morgenstern, O., 1944. Theory of Games and Economic Behavior. Princeton Univ. Press.
Weymark, A., 1991. A reconsideration of the Harsanyi–Sen debate on utilitarianism. In: Elster, J., Roemer, J.E. (Eds.), Interpersonal Comparisons of Well-being. Cambridge Univ. Press.