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Games and Economic Behavior 45 (2003) 132–152

GAMES and Economic Behavior

www.elsevier.com/locate/geb

On authority distributions in organizations: equilibrium

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Received 22 January 2001

Abstract

One assumption in the Shapley–Shubik power index is that there is no interaction nor influence among the voting members. This paper will apply the command structure of Shapley (1994) to model members' interaction relations by simple games. An equilibrium authority distribution is then formulated by the power-in/power-out mechanism. It turns out to have much similarity to the invariant measure of a Markov chain and therefore some similar interpretations are followed for the new setting. In some sense, one's authority distribution quantifies his general administrative power in the organization and his long-run influence on all members. We provide a few applications in conflict resolution, college and journal ranking, and organizational choice.

JEL classification: C68; C71; D58; D72

Keywords: Authority distribution; Simple game; Power index; Markov chain; Command game; Ranking; Planning; Administrative power; Organizational choice

1. Preliminaries

The paper treats an organization as an ensemble of simple games: one "command game" for each member in the organization. The general "authority distribution" would be based on the internal structure in the long run, instead of some specific external tasks. In literature, Simon (1951) defined a member's authority as his right to select actions affecting the

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0899-8256/$ – see front matter
doi:10.1016/S0899-8256(03)00130-1

whole or part of his organization. Aghion and Tirole (1997) analyzed the organizational issue from another aspect.

1.1. Notations

The conventional notation for simple games was introduced by Von Neumann and Morgenstern (1947) and extended by Shapley (1962). As far as our present subject is concerned, we shall distinguish three levels of Boolean abstraction: lower-case italic letters or numerals for *individuals*, italic capitals for *sets of individuals*, and script capitals for *sets of sets of individuals*; thus $c \in C \in \mathcal{C}$. The empty set of individuals, $\emptyset$, is different from the empty set of sets, $\varnothing$. $\subseteq$ is used for set inclusion and $\subset$ for strict set inclusion, and set subtraction will be indicated by $\setminus$ (read "without"). In naming the elements of a set, we shall often employ the vinculum, thus "$\overline{137}$" for "$\{1, 3, 7\}$." The number of elements of a finite set X is denoted $|X|$.

N will conventionally denote the set of all individuals or members in an organization, and $\mathcal{N}$ will denote its power set, i.e., the set of all subsets (*coalitions*) of N. If i is any element of N, then N_i will denote $N \setminus \bar{i}$ and $\mathcal{N}_i$ will denote the power set of N_i. We denote by $\mathcal{S}^+$ the set of all *supersets* of elements of $\mathcal{S}$, by $\mathcal{S}^\cup$ (or $\mathcal{S}^\cap$) the *union* (or *intersection*, respectively) of all elements of $\mathcal{S}$, and by $\mathcal{S}^m$ the smallest $\mathcal{T} \subseteq \mathcal{S}$ such that $\mathcal{T}^+ = \mathcal{S}^+$, or equivalently, the set of all $R \in \mathcal{S}$ that are minimal, i.e., such that no $T \in \mathcal{S}$ satisfies $T \subset R$.

1.2. Simple games

While a *simple game* on N is often represented as the ordered pair $(N, \mathcal{W})$, we denote it by the symbol $\Gamma(N, \mathcal{W})$ where

$$\varnothing \subset \mathcal{W} = \mathcal{W}^+ \subset \mathcal{N}. \tag{1}$$

The first strict inclusion in this definition tells us that (given $\mathcal{W} = \mathcal{W}^+$) N is always in $\mathcal{W}$ and the second tells us that $\emptyset$ is never in $\mathcal{W}$. Hence in a simple game, $\mathcal{W}^m \neq \varnothing$. In the absence of any more specific interpretation, the coalitions in $\mathcal{W}$ will be called *winning* and those in $\mathcal{N} \setminus \mathcal{W}$ will be called *losing*. By the equality in Eq. (1), every superset of a winning coalition is winning while every subset of a losing coalition is losing. Thus in defining a specific simple game it suffices to list just $\mathcal{W}^m$, the set of *minimal winning coalitions*. A player i is called *essential* if $i \in \mathcal{W}^{m\cup}$; otherwise he is a *dummy*. We call him a *dictator* if $\bar{i} = \mathcal{W}^{m\cup}$.

1.3. Shapley–Shubik power indices

For any ordering $i_1 i_2 \cdots i_n$ of all members in the simple game $\Gamma(N, \mathcal{W})$, we consider the increasing sequence of coalitions

$$\emptyset,\ \overline{i_1},\ \overline{i_1 i_2},\ \overline{i_1 i_2 i_3},\ \ldots,\ \overline{i_1 i_2 \cdots i_n} = N.$$

In this sequence, there exists a unique number t such that $\overline{i_1 i_2 \cdots i_{t-1}}$ is losing while $\overline{i_1 i_2 \cdots i_{t-1} i_t}$ is winning; we say that i_t *pivots* in the sequence. Now consider all possible ways to order N. There are $n!$ of them. The *Shapley–Shubik* (*S–S*) *power index* of a player in the simple game is defined as the fraction of orderings in which he pivots. As already shown (Shapley, 1953; Shapley and Shubik, 1954), his power index may be defined as his probability of pivoting when the members are uniformly randomly ordered.

It is easy to see that the S–S power indices of a simple game are nonnegative and sum up to 1. Dictators, and only dictators, have power 1; dummies, and only dummies, have power 0. Players who appear symmetrically in $\mathcal{W}$ have equal power indices, and it is not hard to see that if "j outweighs i" in the sense that replacing i with j never hurts a winning coalition, then j's power index is necessarily not less than i's. If the game is the winning rule for voting a bill and the players satisfy certain symmetry conditions, then player i's S–S index is the chance of the two "critical" situations:

(1) the bill is passed unless he votes against it;
(2) the bill is blocked unless he votes for it.

In either of the critical situations, player i's vote (either YES or NO) determines the result of the bill. Two symmetry conditions (SC-I and SC-II) are stated in Hu (2001). Let $\mathbf{S}$ be the coalition of the players who will vote for (YES) the bill. In general, it is a random coalition to the voting body before the bill is voted; otherwise the bill is not necessary to vote (since the result is known). Clearly $N \setminus \mathbf{S}$ consists of the players who vote against (NO) the bill. The symmetry conditions can be stated as:

- SC-I: ignorance of particular personalities and interdependence. That is, the probability of $\mathbf{S} = T$ depends only on the size of T.
- SC-II: the size of $\mathbf{S}$ is uniformly distributed on $\{0, 1, \dots, n\}$. Therefore, the probability of $|\mathbf{S}| = k$ is $1/(n+1)$ for any $k \in \{0, 1, \dots, n\}$.

In the absence of any more specific information about $\mathbf{S}$, we shall assume the two symmetry conditions when a vote is concerned in the present paper. Actually the two conditions are not essential. As remarked in Section 7, we may relax them to obtain the *asymmetric authority distribution*. As stated before, one assumption in the definition of the S–S index is that every ordering has the same probability $1/n!$. As already shown (Owen, 1971 ; Shapley, 1977; Owen and Shapley, 1989; Cheng, 1994), this assumption is relaxed in a generalization, called the *non-symmetric Shapley–Owen power index*, where the voters are endowed with different ideologies. Another assumption is that the probabilities are independent, i.e., no interaction exists among the voting members. In most real democratic societies, however, there is some degree of political organization, whereby voters influence each other according to the distribution of authority throughout the organization; we will address this influence in the present paper.

A local organizational topology is set up for each member in the form of a simple game in Section 2. In Section 3, an equilibrium authority distribution is derived for organizations in which each member has some, either direct or indirect, impact on others. Section 4 discusses the ultimate influence between players. In Section 5, we shall show that the authority distribution is a generalization to the S–S power index. A few related applications, such as journal ranking and organizational choice, are addressed in Section 6. Omitted proofs can be found in the theory of Markov chains.

2. Local topology of authority

In an organization, some members may have a certain degree of discretionary power; some may even be "free agents," i.e., dictators in their own "command games," accountable to no one else. Others may be merely "cogs in the machinery," i.e., dummies in their own command games. Let us develop these ideas formally.

2.1. Boss and approval sets

Let i denote a generic member of the organization $N = \overline{12\cdots n}$ (with $n > 1$ finite). In general, there will be certain other individuals or, more generally, sets of other individuals, that i must obey, regardless of his own judgment or desires. We call them *boss sets* and denote them collectively by $\mathcal{B}_i$. Thus, if there is an individual b who can "boss" i, this is indicated by $\bar{b} \in \mathcal{B}_i$, not by $b \in \mathcal{B}_i$. Clearly, the empty coalition $\emptyset$ could never be a boss set ($\emptyset \notin \mathcal{B}_i$). Given any boss set $S \in \mathcal{B}_i$, we shall assume that all its supersets in N_i are also boss sets. That is,

$$\varnothing \subseteq \mathcal{B}_i = \mathcal{B}_i^+ \cap \mathcal{N}_i \subset \mathcal{N}_i, \quad \text{all } i \in N, \tag{2}$$

which may be contrasted with Eq. (1). If i is not bossable, then $\mathcal{B}_i = \varnothing$, or $N_i \notin \mathcal{B}_i$.

We shall also associate with each $i \in N$ another collection of coalitions, $\mathcal{A}_i$, called *approval* sets, in $\mathcal{N}_i$. The consent of any one of these approval sets is sufficient to allow i to act, if he wishes. But it is not able to force him to act. So approval sets are not boss sets, and hence $\mathcal{A}_i \cap \mathcal{B}_i = \varnothing$. On the other hand, if $\emptyset \notin \mathcal{A}_i$, then i cannot act alone without an authorization from an approval set in $\mathcal{A}_i$ or an order from a boss set in $\mathcal{B}_i$. Imitating Eq. (2), if S is an approval set, then all its supersets in $\mathcal{N}_i \setminus \mathcal{B}_i$ are also assumed approval sets.

The collection of approval sets $\mathcal{A}_i$ is a guide to the amount of *personal discretion* that i enjoys, if any. At one extreme, if $\mathcal{A}_i = \mathcal{N}_i$ then i is called a *free agent*. He needs no approval (since $\emptyset \in \mathcal{A}_i$), and no one can boss him (since $\mathcal{B}_i = \varnothing$). We shall denote the set of free agents by F. At the other extreme, if $\mathcal{A}_i = \varnothing$, then i has no discretionary power, and we shall call him a *cog*. For an intermediate example of "partial discretion," merely consider a corporation president who is bossable by a $2/3$ majority of the board of directors but is allowed to follow his own judgment as long as he has the support of a simple majority from the board. In the US Legislature, the president's proposal can be vetoed by $2/3$ of the senators and $2/3$ of the representatives.

We let $\mathcal{Z}_i = \mathcal{A}_i \cup \mathcal{B}_i$ be the set of all approval and boss sets of i. However, if $\mathcal{Z}_i = \varnothing$, then i is a cog, but not bossable. The cog takes no action always! We set the convention $\mathcal{Z}_i \supset \varnothing$ for any $i \in N$ such that the organization does not contain any non-bossable cog. Therefore if no coalition can boss i (i.e., $\mathcal{B}_i = \varnothing$), then either i is a free agent (i.e., $\emptyset \in \mathcal{A}_i$) or he needs at least an authorization from some non-empty approval set. On the other hand, N_i bosses i if he is a cog.

Lemma 2.1. $\mathcal{Z}_i = \mathcal{Z}_i^+ \cap \mathcal{N}_i \subseteq \mathcal{N}_i$.

Proof. It suffices to show that $\mathcal{Z}_i \supseteq \mathcal{Z}_i^+ \cap \mathcal{N}_i$. If $S \in \mathcal{Z}_i^+ \cap \mathcal{N}_i$, then there exists a minimal coalition $T \in \mathcal{N}_i$ such that $T \subseteq S$ and $T \in \mathcal{Z}_i$. Now we have three cases:

(i) if $T \in \mathcal{B}_i$, then $S \in \mathcal{B}_i$ by Eq. (2) and so $S \in \mathcal{Z}_i$;
(ii) if $T \in \mathcal{A}_i$ and $S \notin \mathcal{B}_i$, i.e., T is an approval set and S is a superset of T in $\mathcal{N}_i \setminus \mathcal{B}_i$, then $S \in \mathcal{A}_i$ and $S \in \mathcal{Z}_i$;
(iii) if $T \in \mathcal{A}_i$ and $S \in \mathcal{B}_i$, it is trivial that $S \in \mathcal{Z}_i$.

Therefore $S \in \mathcal{Z}_i$ in all cases and we complete the proof. □

2.2. *Command games*

For a coalition $S \subseteq N$ to "command" $i \in N$, either $S \setminus \bar{i}$ can boss i or i can agree to follow $S \setminus \bar{i}$'s collective suggestion. Formally we define i's *command sets* as

$$\mathcal{W}_i \stackrel{\text{def}}{=} \mathcal{B}_i \cup \{S \cup \bar{i}\colon\ S \in \mathcal{Z}_i\}. \tag{3}$$

It is easy to see that $\{S \cup \bar{i}\colon\ S \in \mathcal{Z}_i\} = \mathcal{Z}_i^+ \setminus \mathcal{Z}_i$ and $\mathcal{B}_i^+ = \mathcal{B}_i \cup \{S \cup \bar{i}\colon\ S \in \mathcal{B}_i\}$ from Lemma 2.1 and Eq. (2). Therefore we have the equivalence: $\mathcal{W}_i = \mathcal{B}_i \cup [\mathcal{Z}_i^+ \setminus \mathcal{Z}_i] = \mathcal{B}_i^+ \cup \{S \cup \bar{i}\colon\ S \in \mathcal{A}_i\}$.

Proposition 2.1. $\mathcal{Z}_i^+ \supseteq \mathcal{W}_i \supseteq \mathcal{B}_i^+$ *and* $\mathcal{W}_i = \mathcal{W}_i^+$.

Proof. First of all, $\mathcal{W}_i = \mathcal{B}_i \cup [\mathcal{Z}_i^+ \setminus \mathcal{Z}_i] \subseteq \mathcal{Z}_i \cup [\mathcal{Z}_i^+ \setminus \mathcal{Z}_i] = \mathcal{Z}_i^+$. Secondly, $\mathcal{W}_i \supseteq \mathcal{B}_i^+$ is from the above equivalent definition. To prove $\mathcal{W}_i = \mathcal{W}_i^+$, for $\forall T \in \mathcal{W}_i^+$, there exists a minimal coalition $S \in \mathcal{W}_i$ such that $S \subseteq T$. If $S \in \mathcal{B}_i$ then $S \subseteq T \setminus \bar{i} \in \mathcal{B}_i$ and hence $T \in \mathcal{W}_i$. Or if $i \in S$ and $S \setminus \bar{i} \in \mathcal{Z}_i$, then $i \in T$ and by Lemma 2.1 (together with $S \setminus \bar{i} \subseteq T \setminus \bar{i}$), $T \setminus \bar{i} \in \mathcal{Z}_i$. Therefore $T \in \mathcal{W}_i$. □

It is not hard to see that the following statements are equivalent:

(i) i is a cog;
(ii) $\mathcal{Z}_i^+ = \mathcal{W}_i = \mathcal{B}_i^+$;
(iii) $\mathcal{W}_i = \mathcal{B}_i^+$;
(iv) $\mathcal{Z}_i^+ = \mathcal{W}_i$.

We can also show that $\mathcal{W}_i \setminus \mathcal{Z}_i = \mathcal{W}_i \setminus \mathcal{B}_i = \{\bar{i} \cup T\colon\ T \in \mathcal{Z}_i\}$, and that $\mathcal{Z}_i \setminus \mathcal{W}_i = \mathcal{Z}_i^+ \setminus \mathcal{W}_i = \mathcal{Z}_i \setminus \mathcal{B}_i = \mathcal{A}_i$, the personal discretion.

As $\mathcal{Z}_i \neq \varnothing$ and $\emptyset \notin \mathcal{B}_i$, Proposition 2.1 and Eq. (3) imply that $G_i \stackrel{\text{def}}{=} \Gamma(N, \mathcal{W}_i)$ is a well-defined simple game. We shall call G_i the *command game* for i. The ensemble of all command games $G \stackrel{\text{def}}{=} \{G_i\colon\ i \in N\}$ completely specifies the authority structure or "constitution" of the organization. In particular, if member i is a free agent, then his command game is just $\Gamma(N, \{i\}^+)$; if member j is a cog, then his command game is $\Gamma(N, \mathcal{B}_j^+)$. However, this does not imply that a cog is not an essential player in other members' command games, nor that a free agent is a dictator or is even essential in other members' command games.

The Confucian model of society has a simple organizational form. According to the Confucianism, there are many principles a society should follow, among them:

(P1) The man follows the king;
(P2) The son follows his father;
(P3) The wife follows her husband;
(P4) The king should respect his people.

This authority structure has circular commands with different degrees of discretion. In most of the Chinese history, this orderly organization worked properly. Violations got punished. This also applies for (P4) because the people would exercise their command by overthrowing the king if his evils are insufferable to the vast majority, say 90%, of the governed.

3. Authority distribution

If member i has some power in member j's command game G_j, and j has some power in i's command game G_i at the same time, then how can we define a fair distribution of "power" throughout the organization? We shall call it "authority distribution" to distinguish it from other uses of the word "power." A Markov-chain structure is introduced.

3.1. Authority equilibrium

An *authority distribution* $\pi : N \to [0, 1]$ over an organization (N, G) satisfies $\pi_i \geqslant 0$ for any $i \in N$ and $\sum_{i \in N} \pi_i = 1$. For convenience, we let π be a $1 \times n$ row vector. Denote by $P(i, j)$ member j's S–S power index in member i's command game G_i. And we call the $n \times n$ matrix $P = [P(i, j)]_{i,j=1}^n$ the *power transition matrix* of the organization. If $P(i, j) > 0$, then some of member i's "power," if he has, transfers to member j. $P(i, i)$ measures i's personal discretion $\mathcal{A}_i$. Clearly the power transition matrix P is a *stochastic matrix*, with nonnegative entries and each row sums to 1.

The Shapley value was first derived from three fairness axioms for superadditive games. As far as simple games are concerned, Dubey (1975) modifies one of the axioms and proves the existence and uniqueness of the S–S power index without reference to general (non-simple) games. To state these axioms, we let $\mathcal{C}(N)$ denote the set of all simple games on N, and we use the function $v : 2^N \to \{0, 1\}$ to indicate an element $v \in \mathcal{C}(N)$. That is, $v(S) = 1$ if and only if S is winning. Moreover for simple games $v, w \in \mathcal{C}(N)$, the operations $\vee$ and $\wedge$ are defined by

$$(v \vee w)(S) \stackrel{\text{def}}{=} \max\bigl(v(S), w(S)\bigr), \qquad (v \wedge w)(S) \stackrel{\text{def}}{=} \min\bigl(v(S), w(S)\bigr).$$

As already shown (Dubey, 1975), there exists a unique function $\varphi : \mathcal{C}(N) \to [0, 1]^n$ such that φ satisfies the following axioms:

(A1) for any $v \in \mathcal{C}(N)$, $\sum_{j\in N} \varphi_j(v) = 1$ and $\varphi_j(v) = 0$ if j is a dummy in v;
(A2) if members i and j have symmetric roles in $v \in \mathcal{C}(N)$, then $\varphi_i(v) = \varphi_j(v)$;
(A3) $\varphi(v \vee w) + \varphi(v \wedge w) = \varphi(v) + \varphi(w)$ for any $v, w \in \mathcal{C}(N)$;

and $\varphi_i(v)$ is i's S–S power index in the simple game $v \in \mathcal{C}(N)$. In an organization, we assume that member i's authority is derived from his "command powers" in all members' command games, which assume the Axioms (A1)–(A3). So i's command power in $\mathcal{W}_j$ is just $P(j, i)$. We also note, in reality, that one's command power over those with higher authority makes more contribution to his authority than his command power over those with lower authority, other things equal. For example, the computers in our society have no authority even though they work well; however, a manager has more authority than a programmer, not because the manager knows the computers better, but because he has more command powers over the programmer than that the programmer has over the manager. Formally, we propose that an authority distribution π should follow Axiom (A4).

(A4) $\pi_i = \sum_{j\in N} \pi_j P(j, i)$ for $\forall i \in N$, or in matrix, $\pi = \pi P$.

We call $\pi = \pi P$ the *authority equilibrium equation*. The existence of π is well known from the Markovian theory. In a naive sense of "counterbalance equilibrium," $\pi_j P(j, i)$ is the authority flowing from j to i. So π_i is the sum of those flowing into i: $\pi_i = \sum_{j\in N} \pi_j P(j, i)$. However, in general $\pi_i P(i, j) \neq \pi_j P(j, i)$. For example, let $N = \overline{123}$ where $\mathcal{W}_1 = \{\overline{12}, \overline{13}, \overline{23}\}^+$, $\mathcal{W}_2 = \{\overline{21}, \overline{23}\}^+$, and $\mathcal{W}_3 = \{\overline{23}\}^+$. Then

$$P = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad \text{and} \quad \pi = \left(\frac{1}{7}, \frac{4}{7}, \frac{2}{7}\right);$$

but $\pi_1 P(1, 2) \neq \pi_2 P(2, 1)$.

We should remark that our present interpretation of the "command games" does not envisage "commands" as being instructions that must be absolutely obeyed, as in a military or legal setting. Rather, perhaps, we might here call them "recommendation," "influence," or "advising" games, although we have up to this point excluded the so-called "improper" simple games, in which two disjoint coalitions can both be winning.

However, if there is no clear command-game setting in the organization, it is not hard to formulate the power transition matrix P to designate how much the members "influence each other directly," either by prestige, moral, or else on personal decision making. For example, a 10-year-old child's direct decision may be determined by his parents' guidance with probability 0.3, by peer children with probability 0.4, and his own interest with probability 0.3, although there may be no specific command game on the child. The direct person-to-person impact $P(i, j)$ can also be estimated by linear regressions given lots of historical data. The restriction is that P should be a stochastic matrix.

A concrete organization also holds financial and physical assets, besides its human resource. To include these bossable members into an organization, we introduce the concept of *yesman*. A member i is called a *yesman* if i is a dummy in all command games. Equivalently, the ith column in P is a zero vector. A yesman, such as a slave, shows no sign of personal opinion until his bosses have given him some instructions or hints of the direction. A typical yesman is a computer, which follows all instructions from its users. A secretary sometimes is a yesman, when he is only responsible to his boss sets and takes no positions in all command games. By (A4), the existence of yesmanship in an organization does not affect the authority distribution; and hence we may remove them when calculating the authority distribution. However, yesmen may have some capability, and cost as well, in some tasks with the same capability–cost structure as that of other members; so we can treat the organization's capability–cost in a uniform way. Clearly, a yesman is a cog, but a cog is not necessarily a yesman. On the other hand, a *pooh-bah*, who holds many public or private posts should be in a high rank of authority or of great influence, as indicated in (A4).

3.2. Administrative power

In some sense, π_i is player i's probability of being critical in organizational decisions in the long run. This is our second interpretation for the authority distribution. Say, there is a bill to vote in the organization and we assume that each member has to vote either YES or NO. Then the winning rule for the bill can be formulated as a simple game $\Gamma(N, \mathcal{V})$ on N. As we see in many organizations, different bills may have different voting rules, even within the same organization.

Let q be a probability distribution to pick a player from the organization where $q(i)$ equals to player i's S–S index in the voting game $\Gamma(N, \mathcal{V})$. One could propose a "process" interpretation for the transition probabilities P. In this interpretation, the r.v. X_t denotes the player who is "influential" to the bill in period t. At time $t = 0$, a player is chosen to be the initial influential player according to the probability distribution q. If $X_0 = i_0$, then player i_0 is influential in the sense that the organizational decision is made in period $t = 0$ according to the voting game $\Gamma(N, \mathcal{V})$. Given $X_0 = i_0$, the influential player in period $t = 1$ will be chosen according to i_0's command game. That is, conditional on $X_0 = i_0$, the influential player in period $t = 1$ will be player j with probability $P(i_0, j)$. Hence, $P(i_0, j) = \text{Prob}(X_1 = j \mid X_0 = i_0)$ can be interpreted as the "one-step" influence of player j on player i_0. More generally, $P^m(i, j)$ is interpreted as the m-step influence of player j on player i. Given the initial distribution q, it follows that $\sum_{i \in N} P^m(i, j)q(i)$ is interpreted simply as player j's m-step influence on the bill. In the irreducible and aperiodic case (see Section 3.3), it is well known that $\lim_{m\to\infty} \sum_{i\in N} P^m(i, j)q(i) = \pi_j$ for all initial distribution q. In other words, player j's long-run influence on the bill exists and equals to π_j. This long-run influence does not depend on the specific bill to vote or its winning rule $\Gamma(N, \mathcal{V})$. Furthermore, if we take $q(i) = 1$ for any fixed $i \in N$, then $\pi_j = \lim_{m\to\infty} P^m(i, j)$. This justifies the interpretation for π_j as the long-run influence of player j on other players. We shall address this point in Section 4.

If we reverse the above process, we could obtain another interactive process. Let the bill be proposed in period $t = 0$ and let the proposed bill be formally voted in period $t = \delta$ ($\delta \gg 0$). If the players are not well organized and they vote independently (i.e., all players are free agents), then player i's probability of being critical in the voting game $\Gamma(N, \mathcal{V})$ equals to his S–S index in the game (given the conditions SC-I and SC-II are satisfied). In general, before the period $t = \delta$, every member can exercise his power in others' command games to alter or re-confirm their preference of choices, if he can. And these "exercises of power" make a stochastic point process during the time periods $\{0, 1, 2, \ldots, \delta\}$. We want to find out how frequently each member is critical in the interactive decision-making process. In the absence of any specific pattern of randomness except the two symmetry conditions, we can figure out an embedded and truncated Markov chain $\{Y_0, Y_1, Y_2, \ldots, Y_\delta\}$ where the r.v. Y_t is the critical player in period $\delta - t$.

Without loss of generality, we assume that Y_0 is chosen from N with a probability distribution q'. Therefore, the result of the bill will be determined by Y_0's critical vote in period δ: the bill is passed if Y_0 votes YES or the bill is blocked if Y_0 votes NO. If the organization knows that i_0 will be the critical player in voting the bill, then all players will exercise their powers in i_0's command game in period $\delta - 1$. More formally, in period $\delta - 1$, given $Y_0 = i_0$, there would be a vote to determine i_0's choice (YES or NO) of period δ. The vote has the winning rule $\mathcal{W}_{i_0}$. Of course, i_0 has to obey the decision by his command game and in period δ he will take the choice, which is determined by his command game in period $\delta - 1$. Mathematically, given $Y_0 = i_0$, i_0's choice in period δ is determined by his command game of period $\delta - 1$. If the command game decides the choice YES (or NO) for i_0, then i_0 must vote YES (or NO) in period δ. Now in i_0's command game, member i_1 is critical with probability $P(i_0, i_1)$. We take Y_1 to be the critical member in the vote of period $\delta - 1$. Then Y_1 has the conditional probability distribution: $\mathrm{Prob}(Y_1 = i_1 \mid Y_0 = i_0) = P(i_0, i_1)$. So if Y_1 votes YES (or NO) in period $\delta - 1$, then Y_0 has to vote YES (or NO) in period δ and therefore the bill is passed (or blocked). That is, Y_1 is the real critical member in the last two periods δ and $\delta - 1$. Once again, given that i_1 is critical in period $\delta - 1$, i_1's critical decision is subject to Y_2's critical vote in the command game $\Gamma(N, \mathcal{W}_{i_1})$ of period $\delta - 2$, and Y_2 has the conditional probability distribution $\mathrm{Prob}(Y_2 = i_2 \mid Y_1 = i_1) = P(i_1, i_2)$. Therefore Y_2 is the real critical member in the last three periods $\delta - 2$, $\delta - 1$, and δ. We continue this process and let $\delta \to \infty$; then we obtain a homogeneous Markov chain $\{Y_0, Y_1, \ldots\}$ (see Fig. 1) with probability transition matrix P.

If the process is irreducible and aperiodic, π_i is the frequency of player i's presence in the chain. On the other hand, the distribution π is independent of the specific bill to vote and its respective voting rule; it is also independent of the initial probability distribution q' by which Y_0 is chosen.

In the above senses, we say that π_i is player i's "general administrative power" in the organization.

3.3. Asymptotic properties

The Markov-chain setting of authority distribution shall lead to some similar implications. For any nonnegative integer k, we denote P^k the kth power of P with the convention $P^0 = I$, the $n \times n$ identity matrix. We let $P^k(j, i)$ be the entry at the jth row and the ith

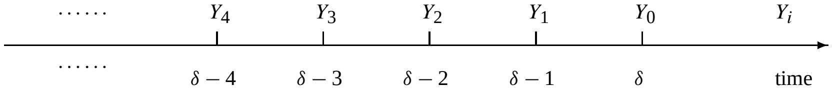

Fig. 1. Interpretation of authority by Markov chains.

column of the matrix P^k, and call i *influences* j (or *influences* j *directly*) if $P^k(j,i) > 0$ (or $P(j,i) > 0$) for some integer $k \geqslant 0$. We denote this as $j \longrightarrow i$ or $i \longleftarrow j$. Otherwise, $j \not\longrightarrow i$ or $i \not\longleftarrow j$. Although $\longrightarrow$ or $\longleftarrow$ is reflexive and transitive, it is not symmetric in general. Two members i and j are said to *communicate* if they influence each other, denoted $i \longleftrightarrow j$. Communication $\longleftrightarrow$ is symmetric. Communication defines an equivalence relationship over the members: two members that communicate with each other are said to be in the same equivalence class. Therefore, any two classes are either disjoint or identical. We say that an organization (or a coalition) is *irreducible* if any two members in the organization (or coalition) communicate with each other. Otherwise, it's *reducible*. In a naive sense, one influences all members and vice versa, in an irreducible organization or coalition. Moreover, if $i \longrightarrow j$ and $j \longleftrightarrow k$, then $i \longrightarrow k$; if $i \longleftrightarrow j$ and $j \longrightarrow k$, then $i \longrightarrow k$. That is, influence is a class property as well. This implies that there is no free agent in an irreducible organization since no one else influences a free agent except himself.

The organization with $N = \overline{123}$, $\mathcal{W}_1 = \{\overline{12}, \overline{13}, \overline{23}\}^+$, $\mathcal{W}_2 = \{\overline{21}, \overline{23}\}^+$, and $\mathcal{W}_3 = \{\overline{23}\}^+$ is irreducible. The organization with $N = \overline{123456}$, $\mathcal{W}_1 = \{\overline{12}, \overline{13}, \overline{15}\}^+$, $\mathcal{W}_2 = \{\overline{12}, \overline{24}\}^+$, $\mathcal{W}_3 = \{\overline{34}\}^+$, $\mathcal{W}_4 = \{\bar{3}, \bar{4}\}^+$, $\mathcal{W}_5 = \{\bar{6}\}^+$, and $\mathcal{W}_6 = \{\bar{5}\}^+$ is reducible. Here $\{3,4\}$ is an irreducible class while $\{5,6\}$ is another irreducible class. However, neither member 1 nor member 2 belongs to any irreducible class. We also note that there exists no influence between the two irreducible classes. In general, the members in an organization N can be classified, in a unique manner, into non-overlapping coalitions $T, C_1, C_2, \ldots, C_k$ such that each C_i is an irreducible class and

$$T = \{i \in N \mid i \longrightarrow j \text{ but } i \not\longleftarrow j \text{ for some } j \in N\}.$$

We call the members of T *authority-transient*. Each C_i has no impact on other irreducible classes, and vice versa. So C_i is actually *"independent"* of all other irreducible classes. For our further convenience, we shall assume that the organization has the above decomposition and therefore P has the form (after some possible re-arrangements) of

$$P = \begin{bmatrix} P_T & Q_1 & \cdots & Q_k \\ & P_{C_1} & & \\ & & \ddots & \\ & & & P_{C_k} \end{bmatrix}, \tag{4}$$

where P_S is the restriction of P to $S \subseteq N$. We let $m = |T|$. It is easy to see that $\lim_{t\to\infty} P_T^t$ is a zero matrix. This implies that:

(i) $I - P_T$ is invertible;
(ii) members in T have no authority, i.e., $\pi_i = 0$ for all $i \in T$; and
(iii) for any organization N, there exists at least one irreducible class.

Influences can also be illustrated by directed graphs. Let the set of vertices $V = N$ and the set of (directed) edges $E = \{(i,j) \in N \times N \mid P(i,j) > 0\}$. A path from u to v is a linked list of vertices $u = u_0, u_1, u_2, \ldots, u_t = v$ such that $(u_i, u_{i+1}) \in E$. In the terminology of graph, $i \longrightarrow j$ if and only if there is a path from i to j. Therefore, an organization is irreducible if and only if there exists a path from i to j for $\forall i, j \in N$.

Member i is said to have *period* d if $P^t(i,i)=0$ whenever $t>0$ is not divisible by d and d is the greatest integer with this property. A member with period 1 is said to be *aperiodic*. Let $d(i)$ denote the period of i. Periodicity is also a class property: if $i \longleftrightarrow j$, then $d(i)=d(j)$. For an example, let $\mathcal{W}_i=\{\overline{i+1,i-1}\}^+$ for $i=2,\ldots,n-1$, $\mathcal{W}_1=\{\overline{n,2}\}^+$, and $\mathcal{W}_n=\{\overline{n-1,1}\}^+$. Then each member has period 2 if n is even or 1 if n is odd. From the theory of Markov chains, for an irreducible organization with period d, we can decompose N as a union of disjoint nonempty coalitions: $N=D_1\cup D_2\cup\cdots\cup D_d$ such that D_i influences D_{i+1} directly (we use the modular notation $d+1=1$). That is, $P(j,k)>0$ for some $j\in D_i$ and $k\in D_{i+1}$. Therefore, in an irreducible organization with period d, we let

$$S_i \stackrel{\text{def}}{=} \left\{ j\in N \mid P^{td}(i,j)>0 \text{ for some } t>0 \right\}$$

for $\forall i\in N$. Then $j\in S_i$ if and only if $i\in S_j$. We list a few asymptotic behaviors here.

Proposition 3.1.

(i) *If i has period $d(i)$ and $P^s(j,i)>0$, then $P^{s+td(i)}(j,i)>0$ for all t sufficiently large.*
(ii) *In an irreducible organization with period d, $\lim_{t\to\infty} P^{td}(i,j)$ exists for $\forall i\in S_j$; furthermore, there is a unique method to assign an authority distribution π to the authority equilibrium equation $\pi=\pi P$, that is, $\pi_j=\frac{1}{d}\lim_{t\to\infty}P^{td}(i,j)$ for $\forall i\in S_j$; and given any initial distribution $\pi^{(0)}$ and define $\pi^{(t+1)}=\pi^{(t)}P$, then $\pi_j=\frac{1}{d}\lim_{t\to\infty}(\pi^{(td)})_j$, the j-th component of the vector $\pi^{(td)}$. Moreover, $\sum_{j\in S_i}\pi_j=\frac{1}{d}$ for $\forall i\in N$. The convergence is linear.*
(iii) *In an irreducible organization, $\pi_i>0$ for $\forall i\in N$.*

Proposition 3.1 (i) says that if i influenced j sometime in the past, then i will influence j periodically in some future. Proposition 3.1 (ii) can be used to approximate the authority distribution by the numerical iterations. However, when there are two or more irreducible classes in the organization, the solution to the authority equilibrium equation $\pi=\pi P$ is not unique. More generally, for the organization with the decomposition as in Eq. (4), $\{\pi\in[0,1]^n \mid \pi=\pi P, \sum_{i\in N}\pi_i=1\}$ is a $k-1$ dimensional convex hull in $[0,1]^n$. One remedy is to assign a specific *authority quota* to each class. Actually, if we assign class C_i with authority quota q_i such that $0\leqslant q_i\leqslant 1$ and $\sum_{i=1}^k q_i=1$, then there exists a unique authority distribution to solve the authority equilibrium equation $\pi=\pi P$ and the quota restriction $\sum_{j\in C_i}\pi_j=q_i$ for all $1\leqslant i\leqslant k$. Another remedy is to consider the *internal authority distribution* within each class only.

3.4. Authority within a class

Let C be an irreducible class and P_C be the restriction of P to C. Formally, we can define the *internal authority distribution* for members within the class C, denoted π^C, by

$$\pi^C=\pi^C P_C, \qquad \sum_{i\in C}\pi_i^C=1, \quad \pi_i^C\geqslant 0,\ \forall i\in C. \tag{5}$$

Proposition 3.2. (i) π^C *exists uniquely*; (ii) *if* C *has period* d *and* $j \in C$, *then* $\lim_{t\to\infty} P^{td}(i,j) = d\pi_j^C$ *for* $\forall i \in S_j$.

Each member in an organization is directly concerned with only a fraction of all the official orders, requisitions, etc. that flow through the organization, so the power transition matrix P will likely be quite sparse—i.e., have lots of zeros. By (A4), the existence of T does not affect the authority distribution. So we may remove T from N before we calculate the authority distribution. This reduction is useful when a large organization has a large portion of authority-transient members. For example, given

$$P = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

after we remove the authority-transient members $T = \overline{12}$, we have

$$P_{N\setminus T} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

So $(\pi_3, \pi_4) = (\pi_3, \pi_4)P_{N\setminus T}$, together with $\pi_1 = \pi_2 = 0$, gives that $\pi_3 = 3/7$, $\pi_4 = 4/7$.

As a remark, given a command game G_i, we can compute its S–S power indices uniquely. But the inverse is not true. For example, in $N = \overline{12345}$, both sets of minimal winning coalitions $\mathcal{W}_1^m = \{\overline{123}, \overline{145}\}$ and $\mathcal{W}_1^m = \{\overline{124}, \overline{135}\}$ have the same S–S power index vector $(7/15, 2/15, 2/15, 2/15, 2/15)$. Moreover, if $\sum_{j\in S} P(i,j) = 1$ for some individual i and coalition S, then S necessarily commands i. However, the inverse is false. In the above example, $\overline{1234}$ commands 1; but $\sum_{j=1,2,3,4} P(1,j) = 13/15 < 1$.

4. Ultimate influence by authority

The S–S power index $P(i,j)$ measures j's direct influence on i. For the "ultimate influence" or "ultimate power," accordingly, is related to $\lim_{t\to\infty} P^t(i,j)$, if it exists. By Proposition 3.1, $\pi_i = \lim_{t\to\infty} P^t(j,i)$ if the organization is irreducible and aperiodic; and the limit is independent of the choice of j. This is our third interpretation of π_i as i's "uniform ultimate influence" to any other members in the organization. Proposition 3.2 provides the ultimate influence among members within the same irreducible class.

In other situations, we notice the less practical importance when both members are authority-transient or when both members belong to different irreducible classes. In these two cases, the ultimate influences between the two members are zeros. In this section, we will go to details for the case of non-transient over transient members.

4.1. Limit influence of a class

For any $i \in T$, there may be some $j \in T$ such that j influences i. But since j has no authority in the organization, his influences on i should come from the members in $N \setminus T$.

$$\boxed{\text{Division I}} \longleftarrow \boxed{\text{Division II}} \longrightarrow \boxed{\text{Division III}}$$

Fig. 2. An example of authority-transient sub-organizations.

In some sense, j acts as “messenger” only, passing the influences from $N \setminus T$ to i. This argument can also be generalized to the case of influence between sub-organizations. In Fig. 2, both Divisions I and III have influence on Division II; however the later has no impact on Division I or III. One question is their relative importance on Division II and by how much.

Given an irreducible class C and any $i \in T$, we propose an authority accumulative sequence $\{\mu_i^C(t)\}_{t=0}^{\infty}$ defined by

$$\mu_i^C(t+1) = \sum_{j \in T} P(i,j)\mu_j^C(t) + \sum_{j \in C} P(i,j)$$

with the initial condition $\mu_j^C(0) = 0$ for $\forall j \in T$. Intuitively $\mu_i^C(t+1)$ is the authority absorption in C from i at or before the $t+1$st step: $\sum_{j \in C} P(i,j)$ is the “authority” absorbed directly by C from i, while $\sum_{j \in T} P(i,j)\mu_j^C(t)$ is the “authority” absorbed indirectly by C from i before the $t+1$st step. Therefore the ultimate authority absorption $\lim_{t\to\infty} \mu_i^C(t)$, if it exists, measures the ultimate influence by C on member i.

Proposition 4.1. *The limit $\mu_i^C = \lim_{t\to\infty} \mu_i^C(t)$ exists and it is the minimal nonnegative solution of the system of inhomogeneous equations*

$$\mu_i^C = \sum_{j \in T} P(i,j)\mu_j^C + \sum_{j \in C} P(i,j). \tag{6}$$

Proof. For any $i \in T$, $\mu_i^C(t)$ is non-decreasing in t (by induction), but it remains bounded by 1. So $\lim_{t\to\infty} \mu_i^C(t)$ exists. The limit obviously satisfies Eq. (6). Conversely, if $\{\mu_i^C\}$ is any nonnegative solution of Eq. (6), we have $\mu_i^C \geqslant \sum_{j \in C} P(i,j) = \mu_i^C(1)$. By induction, $\mu_i^C \geqslant \mu_i^C(t)$ for all $t \geqslant 1$, and so the limits $\lim_{t\to\infty}\{\mu_i^C(t)\}$ represents a minimal nonnegative solution. □

Corollary 4.1. *Given $T, C_1, C_2, \ldots, C_k$ in N as of Eq. (4), for any $i \in T$ and any irreducible class C,*

(i) $\mu_i^C(t) = \sum_{j \in C} P^t(i,j)$;
(ii) $\sum_{j=1}^k \mu_i^{C_j} = 1$, *the irreducible classes have total ultimate influence over T;*
(iii) *if $T = \overline{12\cdots m}$, then*

$$\begin{pmatrix} \mu_1^C \\ \vdots \\ \mu_m^C \end{pmatrix} = (I - P_T)^{-1} \begin{pmatrix} \sum_{t \in C} P(1,t) \\ \vdots \\ \sum_{t \in C} P(m,t) \end{pmatrix}.$$

Proof. Clearly part (i) holds true for $t = 1$. Now if we assume that it is true for some $t \geqslant 1$, then

$$\begin{aligned}\sum_{j\in C} P^{t+1}(i,j) &= \sum_{j\in C}\left[\sum_{s\in T} P(i,s)P^t(s,j) + \sum_{s\in C} P(i,s)P^t(s,j)\right]\\ &= \sum_{s\in T} P(i,s)\sum_{j\in C} P^t(s,j) + \sum_{s\in C} P(i,s)\sum_{j\in C} P^t(s,j)\\ &= \sum_{s\in T} P(i,s)\mu_s^C(t) + \sum_{s\in C} P(i,s)1 = \mu_i^C(t+1).\end{aligned}$$

By the principle of induction, the statement (i) is true for all $t \geqslant 1$. Part (ii) is from (i) and that $\lim_{t\to\infty} P^t(i,j) = 0$ for $\forall i, j \in T$. Part (iii) is from Eq. (6). □

4.2. Redistribution of ultimate influence

The next proposition illustrates how to re-distribute the irreducible class C's total ultimate influence μ_i^C on $i \in T$ among the members of C.

Proposition 4.2. *For any $i \in T$ and any aperiodic irreducible class C, $\lim_{t\to\infty} P^t(i,j) = \pi_j^C \mu_i^C$ for $\forall j \in C$.*

Proof. As stated before, $\lim_{t\to\infty} P^t(i,s) = 0$ for any $i, s \in T$. Now,

$$\begin{aligned}\lim_{t\to\infty} P^{t+1}(i,j) &= \lim_{t\to\infty}\sum_{s\in T} P^t(i,s)P(s,j) + \lim_{t\to\infty}\sum_{s\in C} P^t(i,s)P(s,j)\\ &= \sum_{s\in T}\lim_{t\to\infty} P^t(i,s)P(s,j) + \sum_{s\in C}\lim_{t\to\infty} P^t(i,s)P(s,j)\\ &= \sum_{s\in T} 0P(s,j) + \sum_{s\in C}\lim_{t\to\infty} P^t(i,s)P(s,j)\\ &= \sum_{s\in C}\Big(\lim_{t\to\infty} P^t(i,s)\Big)P(s,j).\end{aligned}$$

We solve the system of equations, contrasted with Eq. (5),

$$\begin{cases}\lim_{t\to\infty} P^t(i,j) = \sum_{s\in C}\Big(\lim_{t\to\infty} P^t(i,s)\Big)P(s,j), \quad \forall j \in C,\\ \sum_{j\in C}\lim_{t\to\infty} P^t(i,j) = \mu_i^C,\end{cases}$$

to get $\lim_{t\to\infty} P^t(i,j) = \pi_j^C \mu_i^C$. □

Corollary 4.2. *In an aperiodic organization, $\lim_{t\to\infty} P^t$ exists.*

In contrast to authority-transient coalitions or sub-organizations, a closed coalition S keeps all authority inside, although it may influence the outside $N \setminus S$. In Fig. 3, the

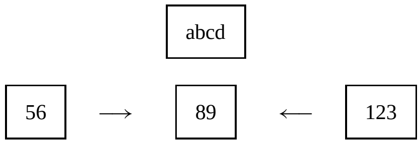

Fig. 3. A closed coalition, $\overline{89}$, keeps all authority inside.

coalition $\overline{89}$ influences $\overline{123}$ and $\overline{56}$. But none of the coalitions $\overline{123}$, $\overline{56}$, or $\overline{abcde}$ influences $\overline{89}$.

We say a coalition S is *authority-closed* if no authority flows out of S, i.e., $P(i, j) = 0$ for all $i \in S$ and $j \in N \setminus S$. However, authority may flow from $N \setminus S$ into S. Examples of closed coalitions include N, F, $\emptyset$, etc. Clearly, irreducible classes are closed. Furthermore, if S_1 and S_2 are closed, then $S_1 \cup S_2$ and $S_1 \cap S_2$ are also closed. When S is closed, then any authority flowing into S will be preserved inside S forever: $P^t(i, j) = 0$, for $\forall i \in S$, $j \notin S$ and $t \geqslant 0$. To see the authority-garnering mechanism, we have that $\sum_{j\in S} P^t(i, j)$ is increasing in t for any closed coalition S. As a consequence, the "total authority" absorbed by a closed S from i, $\lim_{t\to\infty} \sum_{j\in S} P^t(i, j)$ exists.

5. Voting with diverse preference

When we talk about a voting result, we have a scenario of diverse responses. The same result may mean winning for some players to certain degrees; it also means losing for others. Notice, moreover, that the degree to which the result means winning is not the same for the players within the same winning side. For many, the result may even be indifferent. Actually the players have different levels of support for the bill before voting. In this section, we shall show that the S–S power index itself is also an authority distribution associated with the diverse preference of the voters.

Let us return to the simple game $\Gamma(N, \mathcal{W})$ where $N = \overline{12\cdots n}$ being the set of voters and $\mathcal{W}$ being the set of winning coalitions. The voting rule $\Gamma(N, \mathcal{W})$ is set up from the point of "passing the bill." From the viewpoint of the blockers who are against the bill, we can also define "winning" by that "the bill is not passed" or "the bill is blocked" and re-define its "winning" rule Γ^*. A coalition T in Γ^* then has the players who vote against the bill. A "winning" T in Γ^* then makes "passing the bill" impossible, or *blocks* forming any winning coalition in Γ. Therefore it is "winning" in Γ^* if and only if it has nonempty intersection with any winning coalition in Γ. It is not hard to see that Γ^* is also a simple game, denoted $\Gamma^*(N, \mathcal{W}^*)$, where

$$\mathcal{W}^* \stackrel{\text{def}}{=} \{Z \mid Z \cap T \neq \emptyset,\ \forall T \in \mathcal{W}\} = \{N \setminus T \mid T \notin \mathcal{W}\}.$$

The game $\Gamma^*(N, \mathcal{W}^*)$ is called the *dual* of $\Gamma(N, \mathcal{W})$. Given the characteristic function $v(\cdot)$ for Γ, the characteristic function of Γ^* is $v^*(T) \stackrel{\text{def}}{=} 1 - v(N \setminus T)$. It is 1 if $T \in \mathcal{W}^*$ and 0 otherwise.

For the bill to vote, player i has certain level of preference to vote either YES(1) or NO(0), or certain level of intensity to support the bill. It is natural to consider a whole spectrum of preference or intensity from “Absolutely vote NO,” “Strongly Preferred NO,” “Slightly Preferred NO,” “Indifference between YES or NO,” “Slightly Preferred YES,” “Strongly Preferred YES,” to “Absolutely Vote YES,” among other levels. A normal way to quantify the whole spectrum is to map, by one-to-one, the preference levels onto $[0,1]$. We denote i’s preference to voting YES or the intensity to support the bill by p_i. So a p_i of 1, 0.5, or 0 indicates the preference level of “Absolutely vote YES,” “Indifference between YES and NO,” or “Absolutely vote NO,” respectively.

Let us first explain this by the stochastic preference in which p_i denotes player i’s probability to vote for the bill. Denote the vote YES (or NO) by the numerical 1 (or 0, respectively). Say, his vote or vote function $\mathbf{U}_i : [0,1] \to \{0,1\}$ is then determined by his preference p_i in a stochastic way,

$$\mathbf{U}_i = \begin{cases} 1 & \text{with probability } p_i, \\ 0 & \text{with probability } 1 - p_i, \end{cases}$$

i.e., $\mathbf{U}_i$ is a Bernoulli random variable with parameter p_i. Note that there exists no continuous vote function from the continuum of the preference space $[0,1]$ onto the discrete space $\{0,1\}$. Therefore, “player i has more preference to vote YES than player j has” is indicated by $p_i > p_j$, or by $\mathrm{Prob}(\mathbf{U}_i = 1) > \mathrm{Prob}(\mathbf{U}_j = 1)$, or by “$i$ is more likely to vote YES than j,” but not by $\mathbf{U}_i \geqslant \mathbf{U}_j$. This, however, does not exclude the case that “i votes NO and j votes YES.” This case has the probability $(1-p_i)p_j \leqslant \min\{(1-p_i)p_i, (1-p_j)p_j\} \leqslant 0.25$. The more likely cases are that $\mathbf{U}_i \geqslant \mathbf{U}_j$.

More generally, one could explain the stochastic preference by a process. As each bill generally takes some period to be proposed and to be discussed in public before it is formally voted, a player learns the potential advantages and disadvantages to himself if the bill is passed or blocked. He may also be misled by others. In the interactive process, p_i could be player i’s preference at the time when the bill is proposed. Finally, the player has to decide a $\mathbf{U}_i$ for himself. We can imagine that larger p_i generally leads to larger $\mathbf{U}_i$, but not always.

To determine a p_i, one could assume that i expects a (random) payoff $\mathbf{c}_{i,0}$ if the bill is blocked and (random) $\mathbf{c}_{i,1}$ if the bill is passed. Given the lack of the information about other players’ preference, his sincere strategy p_i would maximize his expected utility

$$\max_{p_i \in [0,1]} \mathrm{E}\big[u_i\big(p_i \mathbf{c}_{i,1} + (1-p_i)\mathbf{c}_{i,0}\big)\big],$$

where $u_i(\cdot)$ is i’s utility function. The personal preference p_i can also be obtained by maximizing his satisfaction, diversifying his risk, or reducing his loss, and so on.

Let $\mathbf{S}$ be the random coalition of players who will vote for the bill. If $p_i = 0$, player i’s objective $\mathrm{E}[v(\mathbf{S})] = 0$ is such that there is no chance to pass the bill. If $p_i = 1$, on the other hand, his objective $\mathrm{E}[v(\mathbf{S})] = 1$ is such that the bill will be passed for sure. For a generic player i with $0 < p_i < 1$, he is actually unsatisfactory with both $v(\mathbf{S}) = 0$ and $v(\mathbf{S}) = 1$ with different degrees. His objective is that $\mathrm{E}[v(\mathbf{S})] = p_i$. If this happens, he can represent the group on this issue. In many cases (e.g., multi-person bargaining process or negotiating process), i may propose a new and modified bill to substitute the original one. For example, the bill “Smoking is NOT allowed in public” could be modified as “Smoking is not allowed in a corporate building and its adjacent area within 50 meters; smoking can be allowed in restricted area (smoking area) in restaurants.” For another bill, “If the city has an earthquake of 6.7 degrees, then the state will allocate the city 100 million dollars of emergency relief" can have such modified bills as "If the city has an earthquake of 7.0 degrees, then the state will allocate the city 100 million dollars of emergency relief," or "If the city has an earthquake of 6.7 degrees, then the state will allocate the city 80 million dollars of emergency relief," or "If the city has an earthquake of 6.7 degrees and the earthquake causes 1 billion of loss, then the state will allocate the city 100 million dollars of emergency relief," etc. In general, for any given preference $p \in [0, 1]$, one may come up with many statements which represent p.

For a voter i with $p_i = 0$, he is winning if and only if the bill is blocked. But if the bill is passed, then he has to "obey" the stipulations of the bill regardless of his own choice of vote. We associate the game $\Gamma(N, \mathcal{W})$ as his "command" game: $\Gamma_i = \Gamma$. If $p_i = 1$, then he is winning if and only if the bill is passed. Otherwise he is unsatisfactory with the result and his interest will be affected. We associate the game $\Gamma^*(N, \mathcal{W}^*)$ as his "command" game: $\Gamma_i = \Gamma^*$. Now for any voter i with $0 < p_i < 1$, we take his "command" game Γ_i by some stochastic mechanism or linear interpolation of Γ and Γ^*, say, $\Gamma_i = p_i \Gamma^* + (1 - p_i)\Gamma$. Thus we have set up an organizational structure for the voting body. Note that Γ and its dual Γ^* have the same S–S power indices. By the linearity of the Shapley value, all Γ_i's have the same S–S index as that of Γ. Therefore the power transition matrix P in the structure has $P(i, j) = \varphi_j(v)$, j's S–S index in Γ. Finally we conclude that the S–S index $\{\pi_i = \varphi_i(v)\}_{i=1}^n$ is the unique solution to the authority equilibrium equation $\pi = \pi P$ and it is the authority distribution for the organizational structure. And the solution is independent of the choice of p_i.

6. Applications

In addition to its internal interactions and conflict resolutions, a well organized group N can take external tasks or evaluate external issues as well as it can be controlled or affected by its outsiders. All these aspects would be related to its authority structure or its authority distribution.

6.1. Organizational choice

This subsection works as an application of authority distributions to a simple form of conflict resolutions. Let $p_i^{(t)} \in (-\infty, \infty)$ be member i's (quantitative) opinion on some issue at time t where $t = 0, 1, \ldots$. In the organization, his opinion is assumed to be linearly adopted by all members according to his command powers. That is, his opinion at time $t+1$ is

$$p_i^{(t+1)} = \sum_{j \in N} P(i, j) p_j^{(t)}, \quad \forall i \in N.$$

If we let the column vector $\mathbf{p}^{(t)} = (p_1^{(t)}, p_2^{(t)}, \ldots, p_n^{(t)})$, then $\mathbf{p}^{(t)} = P\mathbf{p}^{(t-1)} = \cdots = P^t \mathbf{p}^{(0)}$.

Proposition 6.1. *In an aperiodic organization with the decomposition Eq.* (4)*, the limit* $\mathbf{p} = \lim_{t\to\infty} \mathbf{p}^{(t)}$ *exists. For any* $j \in C_i$*,* $p_j = \sum_{s\in C_i} \pi_s^{C_i} p_s^{(0)}$*. The limit opinion for* T *is given by* $\mathbf{p}_T = (I - P_T)^{-1} \sum_{i=1}^{k} Q_i \mathbf{p}_{C_i}$*.*

For any $i, j \in T$, although $p_j^{(0)}$ may influence $p_i^{(t)}$ for some finite t, the influence vanishes as $t \to \infty$. The limit p_i for any $i \in T$ is independent of the initial choice of $p_j^{(0)}$, for all $j \in T$. However, Proposition 6.1 does not hold in periodic cases. For example, let $P = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, then we have $\lim_{t\to\infty} \mathbf{p}^{(2t+1)} = \mathbf{p}^{(1)}$; but $\lim_{t\to\infty} \mathbf{p}^{(2t)} = \mathbf{p}^{(0)}$.

Proposition 6.1 provides a simple way to resolve the conflicts within an aperiodic organization. For example, consider an irreducible and aperiodic organization $N = \overline{1234}$ with authority distribution $\pi = (0.3, 0.1, 0.4, 0.2)$. On the issue of the relative utility of one quantity of good x to one quantity of good y, they disagree with each other. Members 1, 2, 3, and 4 initially believe the relative utility is 1.3, 1.4, 1.5, and 1.6, respectively. If the linear adoption scheme applies, the limit opinion would be $1.3*0.3+1.4*0.1+1.5*0.4+1.6*0.2 = 1.45$, which could act as the organizational opinion after long-time discussions or evolutions. For another example, consider an irreducible and aperiodic organization N which wants to admit a few, say t, new employees from m $(m \gg t)$ applicants. To decide who to be admitted, we first put the initial evaluations in an $n \times m$ matrix B, where $B(i, j)$ is member i's initial evaluation on applicant j. So the ith row is member i's initial evaluations on all applicants and the jth column is the initial evaluations on applicant j by all members in the organization. In some real situations, the initial evaluations may be integers between 1 and 9, or on any other scale. If the linear adoption scheme works,

$$\lim_{k\to\infty} P^k B = (\pi_1 \mathbf{1}, \pi_2 \mathbf{1}, \ldots, \pi_n \mathbf{1}) B = \left(\mathbf{1} \sum_{i=1}^{n} \pi_i B(i, 1), \ldots, \mathbf{1} \sum_{i=1}^{m} \pi_i B(i, m) \right),$$

where $\mathbf{1}$ is the column vector with 1's for all entries. Therefore applicant j's eventual or final evaluation (after infinite iterations or discussions) is $\sum_{i=1}^{n} \pi_i B(i, j)$. Then we may give the offers to those who have top t final evaluations.

6.2. Ranking by bilateral impact

To apply the authority distribution to the situations without command games, we analyze a few ranking problems when the data of directly bilateral impact are provided.

6.2.1. College ranking by applicants' acceptance

Suppose that there are large numbers of college applicants to apply the colleges $C_1, C_2, \ldots, C_n$. Each applicant files multiple applications. Each college then offers some of its applicants admissions and rejects all others. Now some applicants may get no offer from any college; the other then get one offer or multiple offers. An applicant with multiple offers will decide which college to go to and reject all other colleges which make offers to him. Of all applicants who apply to and receive offers from C_i, we let $P(i, j)$ be the proportion of those applicants who decide to go to college C_j. Such applicants of course apply to and receive offers from C_j as well. For example, if C_1 admits 10000, of which 8000 decide to register with C_1 and 1500 decide to register with C_2, then $P(1,1) = 8000/10000 = 0.8$ and $P(1,2) = 1500/10000 = 0.15$.

To rank the colleges by the acceptance rates of the applicants to whom offers were made, we can apply the authority distribution associated with the matrix P. We provide an example with the colleges B, H, M, P, S, Y, and O (others). We name them C_1 through C_7, in that order. Assume the acceptance rates have the following matrix P:

	B	H	M	P	S	Y	O
B	0.4	0.1	0.1	0.1	0.1	0.1	0.1
H	0.05	0.9	0.01	0.01	0.01	0.01	0.01
M	0.1	0.2	0.6	0.0	0.05	0.0	0.05
P	0.05	0.1	0.05	0.5	0.05	0.05	0.2
S	0.1	0.2	0.05	0.05	0.45	0.05	0.1
Y	0.05	0.1	0.05	0.05	0.05	0.6	0.1
O	0.05	0.05	0.05	0.05	0.05	0.05	0.7

The so-called "authority distribution" can be regarded as the measure of relative attractiveness of the colleges from the applicants' point of view. The solution of $\pi = \pi P$ is

$$\pi = (0.0868289, 0.530694, 0.073586, 0.0535171, 0.0551895, 0.0654097, 0.134775).$$

Therefore we rank the colleges as: H(1), B(2), M(3), Y(4), S(5), and P(6). If we allow some level, say 0.01, of trust, we may believe that M(0.073586) ties with Y(0.0654097) in the rank 3, and S(0.0535171) ties with P(0.0551895) in the rank 5.

6.2.2. Journal rankings by citations

Let $J_1, J_2, \ldots, J_{n-1}$ be $n-1$ journals in a scientific field, and J_n be the collection of all other journals. We technically treat J_n as a single journal. For any journal J_i, each issue contains many papers, and each paper has its list of references or citations. A paper in J_j can be cited in another paper in J_i as a reference. Of all papers cited by J_i (repetition counted), we let $P(i,j)$ be the proportion of those papers which are published on J_j. So P measures the direct impact between any two journals and $P(i,i)$ is the self-citation rate for J_i. The authority distribution for $\pi = \pi P$ would quantify the long-term influence of each journal in the group of journals and can be used to rank these journals.

6.2.3. Planning of a freeway system

A few small towns believe that building a freeway system would be to their common benefit. Say, they plan to build freeways $F_1, F_2, \ldots, F_{n-1}$. We let F_n be the existing traffic channels of car, truck and bus. We assume that all the potential freeways have the same length. Otherwise we can make up the assumption by dividing long freeways into smaller segments and rename them all. The freeways with higher traffic intensity should be built with more driving lanes and so receive more investments. Of all the traffic flow on F_i, we let $P(i,j)$ be the (estimated) proportion of the traffic flowing into F_j. Then the authority distribution π satisfying $\pi = \pi P$ will measure the relative traffic intensity on each F_i and can be used in the investment allocation.

A similar issue can be found in designing an Internet or Intranet system.

7. Conclusions

In the classical approach to the power distribution for a specific voting problem or a specific voting rule, some set-theoretical methods (simple games) assume only two results, either "winning" or "losing." The S–S index quantifies each member's probability to pivot in an ordering of members. However, an organization is not simply a voting body confronted with a specific voting problem. Our internal structure of "boss" and "approval" is introduced to avoid specific voting problems.

The command games help in providing a conceptual framework for members' specific positions within the organization. Next, the quantitative authority distribution can be used to explain members' general administrative power. It is derived from a political counterbalance equilibrium. A member's authority is from others while his authority is redistributed in his command game. To imitate the decision makings and interactions from the time when a bill is first proposed to the time when it is formally voted, we construct a time-reversed stochastic process of critical players. Given that the critical player in period t is known, the conditional probability of picking the critical player in period $t-1$ obeys the Markovian property. Hence the authority distribution is the frequency of being critical in the long-run decision-making process. In addition, it does not depend on the specific issue to vote when certain conditions, such as irreducibility and aperiodicity, are satisfied. From an imaginative viewpoint, the S–S power index itself is a special authority distribution when the potential negative voting outcomes are concerned. Finally, the distribution can also measure the ultimate influence between members, more precisely between a member and his irreducible organization or sub-organization. In two applications, we use the distribution to resolve internal conflicts and rank journals and colleges.

From a theoretical point of views, there are several related issues worth mentioning. First, to keep the Markovian analysis tractable, we have focused on its interpretation by a decision-making process and its asymptotic behavior. Notice, however, that the set-up is not restrictive for the issues at hand since the theory of Markov chains has rich features of periodicity, convergence, non-homogeneity, generalizations, etc. One could incorporate such features in our model and investigate their rich contexts and interpretations. Secondly, in our analysis, the power transition matrix will always be a stochastic matrix. Hence, any outsider can not interfere with the internal affairs. Hu (2000) has studied some simple forms of campaign strategies to affect an organization's voting by outsiders, based on the same authority structure. Thirdly, commands could also be implemented indirectly. In the related work of Hu and Shapley (2003), a coalition admits its commanded members to command more members. This defines a generalization, called "control game," of command games by iterations of commands. This issue is related to property right, organizational designs, efficiency, and other problems in the theory of organization. Finally, we have constantly mentioned the symmetry conditions (SC-I and SC-II) for the S–S index to quantify a player's chance of being critical. If we replace the S–S indices of P with their respective asymmetric S–S indices, then we would formulate the *asymmetric authority distribution*. In this case, personal preferences on the issue to vote, or other specific factors, are generally taken into account.

Acknowledgments

We thank Hongbin Cai, Shiu-Yuen Cheng, Thomas Ferguson, and Ker-Chau Li for their advice. The authors are also indebted to the referees for providing insightful suggestion and a process interpretation, as well as many corrections.

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