Research Paper No. 2055

Efficient Intertemporal Allocation of Risk and Return

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April 2009

RESEARCH PAPER SERIES

STANFORD
GRADUATE SCHOOL OF BUSINESS

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This work was partially funded by a U.S. National Science Foundation grant, the Center for Advanced Research in Finance at the University of Tokyo, the Frontier Project of the Japanese Ministry of Economics, Trade, and Industry (Grant 07131), Grants-in-Aid for Scientific Research (Grants 208032 and 2053226) from the Japan Society for the Promotion of Science, and the Nomura Foundation for Science and Technology.

ABSTRACT

Efficient allocation of a stochastic stream of financial income is characterized by an explicit stochastic differential equation for the case that each agent has stationary preferences and the probability law of the stochastic process is known. The initial condition is affected by which efficient allocation is chosen, but subsequent evolution is determined solely by agents' impatience and risk aversion.

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EFFICIENT INTERTEMPORAL ALLOCATION OF RISK AND RETURN

ROBERT WILSON AND EIICHIRO KAZUMORI

Abstract. Efficient allocation of a stochastic stream of financial income is characterized by an explicit stochastic differential equation for the case that each agent has stationary preferences and the probability law of the stochastic process is known. The initial condition is affected by which efficient allocation is chosen, but subsequent evolution is determined solely by agents' impatience and risk aversion.

1. Introduction

We consider a financial contract that allocates a stochastic stream of income among several agents. We assume that each agent has stationary preferences for income, but agents can differ in impatience and risk aversion. Due to these differences, each agent can benefit from trading his own income stream for a share of their aggregate income stream. We study a contract that is efficient, i.e. Pareto optimal among the agents, by combining previous analyses by Gollier and Zeckhauser [3, 2005] and Wilson [5, 1968].[1]

We characterize an efficient contract by a stochastic differential equation. This equation specifies how the allocation among agents evolves as the path of aggregate income is realized with the passage of time. The allocation has a simple property — after an initial division of the gains from trade, the evolution of the allocation is determined solely by agents' aversions to delay and risk.

The main ideas are exposited initially without addressing the technical aspects of stochastic processes. Section 2 sets up the formulation, and Section 3 illustrates some special cases.

Date: April 2009. This is an interim report on an ongoing project. Copies can be downloaded from `gsbapps.stanford.edu/researchpapers/library/RP2055.pdf`.

Key words and phrases. finance, contract, efficient allocation, risk aversion, impatience, stochastic differential equation.

JEL subject classifications: D53, D61, D70, D86, D91.

Wilson acknowledges support from the National Science Foundation of the United States. Kazumori acknowledges support from the Center for Advanced Research in Finance at the University of Tokyo, the Frontier Project of the Japanese Ministry of Economics, Trade, and Industry (Grant 07131), Grants-in-Aid for Scientific Research (Grants 208032 and 2053226) from the Japan Society for the Promotion of Science, and the Nomura Foundation for Science and Technology.

[1]See also Becker [1, 1980]. Wilson [5] considers also the case that agents have differing probability assessments about the aggregate income stream, but we omit this extension here. We do not here examine other extensions to cases involving private information and/or learning, imperfect observability, moral hazard, or investments undertaken during the contract.

Then Section 4 states the main result and sketches the proof, which is then completed with a formal treatment in Section 5. Section 6 describes implementation as a market equilibrium.

2. Formulation

We begin by stating our maintained assumptions.

A contract applies to a time interval $T = \{t \mid t_* \leqslant t \leqslant t^*\} \subset \mathbb{R}$, where time is continuous and $t_* < t^*$.[2] Over this interval, aggregate income evolves as a stochastic process according to a known probability law.[3] Let $\mathbb{Y}(t)$ be the set of possible realizations of aggregate income at time t, and assume that $\mathbb{Y} : T \twoheadrightarrow \mathbb{R}$ is a continuous correspondence whose images are intervals. Denote its graph by $\mathrm{Gr}(\mathbb{Y})$ and the interior by $\mathrm{Gr}^\circ(\mathbb{Y})$. A realized path of aggregate income is denoted $Y = (y(t))_{t_* \leqslant t \leqslant t^*}$, where $y(t) \in \mathbb{Y}(t)$ is the aggregate income realized at time t. A partial history up to time t is denoted $\tilde{y}(t)$, i.e. $\tilde{y}(t) = (y(\tau))_{t_* \leqslant \tau \leqslant t}$.

There are N agents denoted by $i = 1, \ldots, N$. Each agent i's instantaneous utility for income is at every time the same function $u_i : A_i \to \mathbb{R}$ that is strictly increasing, strictly concave, and twice differentiable on an open interval $A_i = (a_i, \infty) \subset \mathbb{R}$. In particular, his risk tolerance $c_i(x) = -u_i'(x)/u_i''(x)$ is positive. We assume that $u_i(x) \downarrow -\infty$ as $x \downarrow a_i$, and if $a_i \neq -\infty$ then his risk tolerance $c_i(x) \downarrow 0$ as $x \downarrow a_i$. To ensure existence of feasible allocations, assume that $\sum_i a_i < y(t)$ for every possible realization $y(t)$ of aggregate income at every time $t \in T$.

Agent i's realized utility $U_i(X_i)$ for a Lebesgue measurable income stream $X_i = (x_i(t))_{t_* \leqslant t \leqslant t^*}$ is obtained by discounting, and thus has the form

$$U_i(X_i) = \int_{t_*}^{t^*} e^{-R_i(t)} u_i(x_i(t))\, dt\,.$$

Assume that each instantaneous impatience parameter $R_i(t)$ is positive and the function $R_i : T \to \mathbb{R}$ is differentiable with derivative $r_i(t) = R_i'(t)$.

A contract specifies for each time t and partial history $\tilde{y}(t)$, an allocation $x(t) = (x_i(t))_{i=1,\ldots,N} \in \mathbb{R}^N$ of the current aggregate income $y(t)$ among the agents. Feasibility requires $\sum_i x_i(t) = y(t)$. Moreover, the allocation at time t must be the same for any two realizations Y, Y' of aggregate income with the same partial history $\tilde{y}(t) = \tilde{y}'(t)$. Thus a contract is characterized by agents' realized income streams $(X_i)_{i=1,\ldots,N}$ for each possible realization Y of the stream of aggregate income, and the set of feasible contracts comprises those for which each X_i is measurable with respect to partial histories, each $x_i(t) \in A_i$, and $\sum_i x_i(t) = y(t)$ at each time t for each partial history $\tilde{y}(t)$.

[2] In general the terminal time t^* might be infinite or random but here we assume it is fixed and finite.

[3] Appendix A states technical assumptions about this stochastic process.

A contract is efficient if there exist positive welfare weights $(w_i)_{i=1,\dots,N}$ such that it maximizes the weighted aggregate expected welfare $\sum_i w_i E[U_i(X_i)]$ among all feasible contracts, where $E[\cdot]$ indicates expectation with respect to the probability distribution of streams of aggregate income.

3. Two Special Cases

To introduce the main result proved in Section 5, we first review two special cases addressed in [3, 5].

3.1. **Static Allocation of Aggregate Income.**

A familiar property of static models of risk sharing is that each agent obtains a marginal share that is proportional to his risk tolerance [5]. We sketch here how one obtains this same result when agents have the same impatience, say $r_i = \bar{r}$. In this case it suffices to consider each time $t \in T$ separately.

Regardless of prior history, an efficient contract chooses an allocation $x(y)$ of each realized aggregate income $y \in \mathbb{Y}(t)$ at time t that maximizes $\sum_i w_i E[u_i(x_i(y))]$ subject to each $x_i(y) \in A_i$ and $\sum_i x_i(y) = y$ for every possible realization y of current income. If the constraint $x_i(y) \in A_i$ is not binding then the necessary and sufficient condition for this maximization is that, for each aggregate income y, $w_i u_i'(x_i(y))$ is the same for every agent i. If the efficient allocation is differentiable then this condition implies that for some function $\nu : \mathbb{Y} \to \mathbb{R}$ that is the same for all i,

$$\nu(y) = -\frac{d}{dy} \ln[w_i u_i'(x_i(y))] = c_i(x_i(y))^{-1} \frac{d}{dy} x_i(y)$$

for all i. Therefore $\nu(y) \sum_i c_i(x_i(y)) = \sum_i \frac{d}{dy} x_i(y) = 1$ so

$$\frac{d}{dy} x_i(y) = \frac{c_i(x_i(y)}{\sum_j c_j(x_j(y))} .$$

In the notation of stochastic calculus, this is customarily written as

$$dx_i(y) = \left[\frac{c_i(x_i(y)}{\sum_j c_j(x_j(y))} \right] \cdot dy .$$

Example 1: If each agent i's utility is exponential with constant risk tolerance c_i then

$$x_i(y) = x_i^* + s_i y \qquad \text{where} \quad s_i = c_i / \sum_j c_j$$

and only x_i^* depends on the welfare weights.

Example 2: If each agent i's utility is logarithmic with risk tolerance $c_i(x) = x - a_i$ then

$$d\ln[x_i(y) - a_i] = d\ln[y - \bar{a}] \qquad \text{where} \quad \bar{a} = \sum_i a_i\,.$$

3.2. **Pure Intertemporal Allocation.** A second special case occurs when aggregate income is constant, say $y(t) = \bar{y}$ at every time t. In this case the only motive for the contract is to obtain gains from trade due to differences among agents' impatience, as studied by Gollier and Zeckhauser [3].

For simplicity, assume that $R_i(t) = r_i t$. Then a contract $x : T \to \mathbb{R}^N$ is efficient if it maximizes

$$\sum_i w_i \int_{t_*}^{t^*} e^{-r_i t} u_i(x_i(t))\,dt\,,$$

subject to the feasibility constraints that $\sum_i x_i(t) = \bar{y}$ for all $t \in T$. The necessary and sufficient condition is that $w_i e^{-r_i t} u_i'(x_i(t))$ is the same for every agent i. As in the previous case, if the allocation is differentiable then this condition implies that for some function $\mu : T \to \mathbb{R}$ that is the same for all i,

$$\mu(t) = -\frac{d}{dt}\,\ln[w_i e^{-r_i t} u_i'(x_i(t))] = r_i + c_i(x_i(t))^{-1}\frac{d}{dt}x_i(t)$$

for all i. Therefore, since $\sum_i \frac{d}{dt}x_i(t) = 0$,

$$\mu(t)\sum_i c_i(x_i(t)) = \sum_i r_i c_i(x_i(t))\,,$$

so

$$\frac{d}{dt}x_i(t) = c_i(x_i(t))[\bar{r} - r_i] \qquad \text{where} \quad \bar{r} \equiv \frac{\sum_i r_i c_i(x_i(t)}{\sum_i c_i(x_i(t))}\,.$$

In the notation of stochastic calculus, this is customarily written as

$$dx_i(t) = \Big[c_i(x_i(t))[\bar{r} - r_i]\Big] \cdot dt\,.$$

Example 3: If each agent i's utility is exponential with constant risk tolerance c_i then

$$x_i(t) = x_i(t_*) + c_i[\bar{r} - r_i][t - t_*] \quad \text{where} \quad \bar{r} = \frac{\sum_i c_i r_i}{\sum_i c_i}$$

and only $x_i(t_*)$ depends on the welfare weights.

Example 4: If each agent i's utility is logarithmic with risk tolerance $c_i(x) = x - a_i$ then

$$d\ln[x_i(t) - a_i] = [\bar{r} - r_i] \cdot dt\,,$$

$$\ln(x_i(t) - a_i) = [\bar{r} - r_i][t - t_*] + \ln(x_i(t_*) - a_i)\,,$$

where $\sum_i x_i(t_*) = \bar{y}$.

3.3. **Combinations of the Two Special Cases.** Combining Examples 1 and 3, if agents' risk tolerance coefficients c_i are constants then also $\bar{c}, \bar{r}, s_i$ are constants and therefore

$$x_i(t, y(t)) = \hat{x}_i + c_i[\bar{r} - r_i][t - t_*] + s_i y(t)\,,$$

where the initial allocation satisfies $\sum_i \hat{x}_i = 0$. In this case the induced risk tolerance of each agent is $\bar{c}$, and the induced interest rate of each agent is $\bar{r}$, in the sense that for every $y(t)$ each agent i's instantaneous discounted utility conditional on an efficient allocation is

$$e^{-r_i t} u_i(x_i(t, y(t))) = |u_i(\hat{x}_i)| \times e^{-\bar{r}t}\bar{u}(y(t))\,,$$

where $\bar{u}$ has the constant risk tolerance $\bar{c}$, and only the scale factors $|u_i(\hat{x}_i)|$ reflect the welfare weights used to determine a specific allocation.

Combining Examples 2 and 4, if each $u_i(x) = \log(x - a_i)$ then $c_i(x) = x - a_i$ and the formula for increments can be written as

$$du_i(x_i(t, y(t))) = [\bar{r} - r_i] \cdot dt + d\bar{u}(y(t))\,,$$

where $\bar{u}(y) = \log(y - \bar{a})$ and $\bar{a} = \sum_i a_i$.

4. A Candidate Solution

In this section we address the general case specified in Section 2 and derive a candidate for an efficient contract. Section 5 then establishes that this candidate contract is indeed efficient.

We derive the candidate contract by assuming that it is described by a function $x : \mathrm{Gr}(\mathbb{Y}) \to A$ that to each time t and possible aggregate income $y \in \mathbb{Y}(t)$ assigns an allocation $x(t, y) \in A$, where $A = \prod_i A_i \subset \mathbb{R}^N$. Moreover, we assume that x is a differentiable function on $\mathrm{Gr}^\circ(\mathbb{Y})$.

We use the following notation. For each agent i,

$$\bar{c}(t, y) = \sum_i c_i(x_i(t, y))$$

$$s_i(t, y) = c_i(x_i(t, y))/\bar{c}(t, y)$$

$$\bar{r}(t, y) = \sum_i r_i(t) s_i(t, y)\,.$$

We omit arguments of these functions in the sequel. Note that $\bar{c}$ measures aggregate risk tolerance, s_i is agent i's share of aggregate risk tolerance, and $\bar{r}$ is the weighted-average impatience using agents' shares s_i as weights.

Proposition 4.1. *If x is the allocation rule for an efficient contract then on $Gr^\circ(\mathbb{Y})$ each agent i's allocation satisfies the stochastic differential equation*

$$dx_i = c_i[\bar{r} - r_i] \cdot dt + s_i \cdot dy$$

.

Sketch of proof. An efficient contract maximizes

$$\sum_i w_i E\left[\int_{t_*}^{t^*} e^{-R_i(t)} u_i(x_i(t, y(t)))\, dt\right],$$

for some positive welfare weights $(w_i)_{i=1,\dots,N}$, subject to measurability with respect to partial histories and the feasibility constraints

$$(\forall\,(t,y) \in \mathrm{Gr}(\mathbb{Y})) \qquad x_i(t,y) \in A_i \quad \text{and} \quad \sum_i x_i(t,y) = y\,.$$

Because the maximand is additively separable across times and realizations, this maximization can be done pointwise.[4] If x is a maximizer then there exists a Lagrange multiplier $\lambda(t,y)$ on the feasibility constraint such that for a.e. $(t,y) \in \mathrm{Gr}^\circ(\mathbb{Y})$

$$w_i e^{-R_i(t)} u_i'(x_i(t,y)) = \lambda(t,y)\,,$$

or equivalently, $w_i e^{-R_i(t)} u_i'(x_i(t,y))$ is the same for every agent i. We show that this condition implies that on $\mathrm{Gr}^\circ(\mathbb{Y})$

$$\frac{\partial}{\partial t} x_i = c_i[\bar{r} - r_i] \qquad \text{and} \qquad \frac{\partial}{\partial y} x_i = s_i\,,$$

and therefore

$$\begin{aligned} dx_i &\equiv \frac{\partial}{\partial t} x_i(t,y) \cdot dt + \frac{\partial}{\partial y} x_i(t,y) \cdot dy \\ &= c_i[\bar{r} - r_i] \cdot dt + s_i \cdot dy\,, \end{aligned}$$

which is the stochastic differential equation stated in the proposition.

Because $w_i e^{-R_i(t)} u_i'(x_i(t,y))$ is the same for all i, there exist functions $\mu, \nu : \mathrm{Gr}^\circ(\mathbb{Y}) \to \mathbb{R}$ such that

$$\begin{aligned} \mu(t,y) &= -\frac{\partial}{\partial t} \ln(w_i e^{-R_i(t)} u_i'(x_i(t,y))) = r_i + c_i^{-1} \frac{\partial}{\partial t} x_i \\ \nu(t,y) &= -\frac{\partial}{\partial y} \ln(w_i e^{-R_i(t)} u_i'(x_i(t,y))) = c_i^{-1} \frac{\partial}{\partial y} x_i\,. \end{aligned}$$

[4]See Section 5 for elaboration.

Now the feasibility constraints imply that

$$\sum_i \frac{\partial}{\partial t} x_i = 0, \qquad \text{so} \qquad \mu = \frac{\sum_i r_i c_i}{\sum_i c_i}$$

$$\sum_i \frac{\partial}{\partial y} x_i = 1, \qquad \text{so} \qquad \nu = \frac{1}{\sum_i c_i}\,,$$

and therefore

$$\frac{\partial}{\partial t} x_i = c_i[\bar{r} - r_i] \qquad \text{and} \qquad \frac{\partial}{\partial y} x_i = s_i\,,$$

where $\bar{r} = \mu$ and $s_i = c_i \nu$ as required. □

5. Validation of the Stochastic Differential Equation

This section provides further technical details that verify that the candidate solution in Section 4 is actually an optimal solution, and validates its interpretation as a stochastic differential equation.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t_* \leqslant t \leqslant t^*}, \mathbf{P})$ be a filtered complete probability space satisfying the usual conditions. Assume that the set $\{Y\} = \{y(t)_{t_* \leqslant t \leqslant t^*}\}$ of realizations of aggregate income streams is an $\{\mathcal{F}_t\}$-adapted process, i.e. $\{y(t)_{t_* \leqslant t}\}$ is measurable with respect to $\mathcal{F}_t$. Thus $\mathbf{P}$ induces a probability law for the stochastic process that generates streams of aggregate income.

In general, a contract specifies the agents' consumption process $X = \{x_i(t,\omega)\}_{i=1,\ldots,N}^{t_* \leqslant t \leqslant t^*}$ subject to the feasibility condition that $X \in \mathcal{X}(Y)$ for each realization Y of the aggregate income stream, where $\mathcal{X}(Y)$ is the set of consumption streams for which each allocation $x : [t_*, t^*] \times \Omega \to \mathbb{R}^N$ is measurable and $\{\mathcal{F}_t\}$-adapted, and for each time t and state ω, $\sum_{i=1}^N x_i(t,\omega) = y(t,\omega)$, where $y(t,\omega) \in \mathbb{Y}(t)$ is the realization of aggregate income at time t. If the contract specifies the consumption process X then the realized social welfare is

$$W(X,\omega) = \sum_{i=1}^N w_i \int_{t_*}^{t^*} e^{-R_i(t)} u_i(x_i(t,\omega))\, dt\,,$$

if the state is ω. Denote expected welfare conditional on the aggregate income stream by $W(X,Y) = E[W(X,\omega)|Y]$, and by $W(X) = E[W(X,\omega)]$ overall.

We proceed as follows. First we formulate a relaxed problem for which it is assumed that an omniscient contract designer knows the realization of the state $\omega \in \Omega$, and thus the realization of the aggregate income stream Y. We derive a Bellman equation for the relaxed problem. Then we observe that the solution of the relaxed problem does not depend on the state, so the solution is optimal even when the designer does not know the state.

Now suppose the designer knows the state ω and thus the realization Y of the aggregate income stream. Then conditional on the state ω the designer's problem is to maximize $W(X, Y)$ subject to $X \in \mathcal{X}(Y)$, where now each $x_i(t, \omega)$ is conditioned on the known state ω. For each time $s \in [t_*, t^*]$ let X_s and Y_s be the consumption and income streams from time s onward, and let $\mathcal{X}_s(Y_s)$ be those consumption streams that satisfy the feasibility constraints from time s onward. Then welfare from time s onward is

$$W_s(X_s, Y_s) = \sum_{i=1}^{N} w_i E[\int_s^{t^*} e^{-R_i(t)} u_i(x_i(t, \omega))\, dt | Y_s]\,.$$

Define the value function

$$V_s(Y_s) = \sup_{X_s \in \mathcal{X}_s(Y_s)} W_s(X_s, Y_s)\,.$$

Since u_i is uniformly continuous, the stochastic version of the Principle of Optimality holds.[5] That is, for any $t_* \leqslant s \leqslant \hat{s} \leqslant t^*$,

$$V_s(Y_s) = \sup_{x(t,\omega)_{s \leqslant t \leqslant \hat{s}} \in \mathcal{X}_s(Y_s, [s, \hat{s}])} \sum_{i=1}^{N} w_i E[\int_s^{\hat{s}} e^{-R_i(t)} u_i(x_i(t, \omega))\, dt | Y_s] + V_{\hat{s}}(Y_{\hat{s}})\,,$$

where $\mathcal{X}_s(Y_s, [s, \hat{s}])$ is the projection of $\mathcal{X}_s(Y_s)$ to the initial time interval $[s, \hat{s}]$. Furthermore, if $\overline{x}_i(t, \omega)_{s \leqslant t \leqslant t^*}$ solves the problem from time s on, then

$$V_s(Y_s) = \sum_{i=1}^{N} w_i E[\int_s^{t^*} e^{-R_i(t)} u_i(\overline{x}_i(t, \omega))\, dt | Y_s]\,.$$

Because prior consumption $x(s, \omega)_{t_* \leqslant s < t}$ does not affect future welfare nor the subsequent evolution of aggregate income, we have a following candidate for the solution $x^*(t, \omega)_{s \leqslant t \leqslant \hat{s}}$ of the relaxed problem. For each $t \in [s, \hat{s}]$

$$x^*(t, \omega) \in \arg\max \sum_{i=1}^{N} w_i e^{-R_i(t)} u_i(x_i(t, \omega)) \text{ subject to } \sum_{i=1}^{N} x_i(t, \omega) = y(t, \omega)\,.$$

Since u_i is strictly concave and twice differentiable, and also $u_i(x) \downarrow -\infty$ as $x \downarrow a_i$ and if $a_i \neq -\infty$ then $c_i(x) \downarrow 0$ as $x \downarrow a_i$, the candidate solution $x^*(t, \omega)$ exists and is in $\prod A_i$.

We now check that $x^*(t, \omega)_{s \leqslant t \leqslant \hat{s}}$ solves the Bellman equation. Clearly $x^*(t, \omega)_{s \leqslant t \leqslant \hat{s}}$ satisfies the measurability constraint and the allocation constraint, so it is in $\mathcal{X}_s(Y_s, [s, \hat{s}])$. Suppose it is not optimal and there exists $x'(t, \omega)_{s \leqslant t \leqslant \hat{s}}$ that gives a strictly higher value. Since $V_{\hat{s}}(Y_{\hat{s}})$

[5] Krylov [4, 2009] and Fleming and Soner [2, 2005].

is independent of $x(t,\omega)_{s\leqslant t<\hat{s}}$, this implies that

$$\sum_{i=1}^{N} w_i E[\int_s^{\hat{s}} e^{-R_i(t)} u_i(x_i'(t,\omega))\,dt|Y_s] > \sum_{i=1}^{N} w_i E[\int_s^{\hat{s}} e^{-R_i(t)} u_i(x_i^*(t,\omega))\,dt|Y_s]\,.$$

By taking $\hat{s}\downarrow s$, this implies that

$$\sum_{i=1}^{N} w_i E[e^{-R_i(s)} u_i(x_i'(s,\omega))|Y_s] > \sum_{i=1}^{N} w_i E[e^{-R_i(s)} u_i(x_i^*(s,\omega))|Y_s]\,,$$

which contradicts the definition of $x^*(s,\omega)$. Similarly, one verifies that $x^*(t,\omega)_{t_*\leqslant t\leqslant t^*}$ is an optimal policy for the relaxed problem.

It remains to show that x^* is indeed a solution of the problem where the designer does not know the realization of the state ω. For that purpose note that, since the solution $x^*(t,\omega)_{t_*\leqslant t\leqslant t^*}$ depends only on $\{y(t,\omega)\}_{t_*\leqslant t\leqslant t^*}$ and does not depend on the state ω more than it depends on the realized aggregate income stream $\{y(t,\omega)\}_{t_*\leqslant t\leqslant t^*}$, the designer does not require knowledge of the state ω to implement the solution. Therefore, an efficient contract is the allocation $x^\circ(t,y)_{t_*\leqslant t\leqslant t^*}$ that for each time t and aggregate income y is obtained as the optimal allocation $x^*(t,\omega)_{t_*\leqslant t\leqslant t^*}$ that is the same for each state ω for which $y(t,\omega)$ is the realized aggregate income y.

6. Implementation via a Market

We conclude by showing that each efficient allocation can be supported by an Arrow-Debreu equilibrium where agents can trade arbitrary future payoff streams at the initial date to achieve the given allocation.

We consider a pure exchange economy for a single consumption good. Assume that there exists a market where shares of aggregate income are traded. Holding one share of the stock from $t=t_*$ to $t=t^*$ yields a payoff at rate $y(t)$ at time t. Assume further that there exists a money market in which a locally risk-free security can be traded. Let $p(t)$ be the price of the stock (ex-dividend) and let $r(t)$ be the risk-free interest rate at time t.

There are N agents denoted by $i=1,\ldots,N$. Each agent trades competitively in the money and securities markets and consumes the proceeds. Let $x_i(t)$ be agent i's consumption rate at time t, let $a_i(t)$ be his holdings of the risk-free asset, and let $\theta_i(t)$ be his holdings of the stock. Assume that the consumption and trading strategies $\{x_i(t),a_i(t),\theta_i(t)\}$ are adapted processes with the standard integrability conditions:

$$\int_{t_*}^{t^*} x_i(t)^2\,dt<\infty, \int_{t_*}^{t^*} |a_i(t)r(t)\,dt+\theta_i(t)(y(t)\,dt+dp(t))|<\infty, \int_{t_*}^{t^*} \theta_i(t)^2 d[p(t)]<\infty$$

where $[p(t)]$ is the quadratic variation process of $p(t)$. Each agent's wealth process is $W_i(t) = a_i(t) + \theta_i(t)p(t)$. $W_i(t)$ follows a stochastic differential equation

$$dW_i(t) = a_i(t)r(t)\,dt + \theta_i(t)(y(t)\,dt + dp(t)) - x_i(t)\,dt\,.$$

To exclude arbitrage opportunities, assume that $W_i(t)$ must be positive with probability 1.

Agent i chooses his consumption and trading strategy $\{x_i(t), a_i(t), \theta_i(t)\}$ to maximize his expected utility

$$\int_{t_*}^{t^*} e^{-R_i(t)} u_i(x_i(t))\,dt$$

where the instantaneous utility is strictly increasing, strictly concave, and twice differentiable on an open interval $A_i = (a_i, \infty)$.

A competitive equilibrium of the economy is a pair of price processes $\{p, r\}$ and agents' consumption-trading strategies $\{x_i(t), a_i(t), \theta_i(t)\}_{i=1,\ldots,N}$ such that $\{x_i(t), a_i(t), \theta_i(t)\}$ maximizes the expected utility

$$E[\int_{t_*}^{t^*} e^{-R_i(t)} u_i(x_i(t))\,dt]$$

subject at every time t to

$$dW_i(t) = a_i(t)r(t)\,dt + \theta_i(t)(y(t)\,dt + dp(t)) - x_i(t)\,dt,$$

and markets clear:

$$\sum_{i=1}^{N} \theta_i(t) = 1, \qquad \sum_{i=1}^{N} a_i(t) = 0\,.$$

To derive an equilibrium one first derives the efficient allocation of the economy when a designer computes the optimal sharing rules. This is done as in Proposition 4.1 and Section 5. For any efficient allocation, an Arrow-Debreu equilibrium can be derived that supports the allocation. In an Arrow-Debreu equilibrium, agents trade arbitrary payoff streams at the initial date. The equilibrium is defined by the pricing function $\{\phi_{t_*}(s), t_* \leqslant s \leqslant t\}$ such that the price of an arbitrary payoff stream $\{x(s), t_* \leqslant s \leqslant t\}$ at $t = t_*$ is given by the linear functional $\Phi_{t_*}(x) = E_{t_*}[\int_{t_*}^{t^*} \phi_{t_*}(s)x(s)\,ds]$, and the market clears. By the second welfare theorem, one knows that for each w and the corresponding efficient allocation $x_i^*(t)$, there exists an Arrow-Debreu equilibrium that leads to the same allocation.

In general, $\phi_{t_*}(s)$ can depend on t_* and the whole time path of Y up to time s, which gives the relevant description of the underlying state of the economy at time s. In the current setting, however, due to time-additive and state-separable preferences of the agents, $\phi_{t_*}(s)$

depends only on w, $y(t_*)$, and $(s, y(s))$. Thus the Arrow-Debreu price of a payoff stream $x(s)_{t_* \leqslant s \leqslant t}$ up to any time t is given by $\Phi_{t_*}(x) = E[\int_{t_*}^{t} \phi_{t_*}(s) x(s)\, ds]$.

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